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On the Skein Theory of 0-framed Surgery Along the Trefoil Knot

Andrew Robert Holmes

Louisiana State University and Agricultural and Mechanical College, aholm27@lsu.edu

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ON THE SKEIN THEORY OF 0-FRAMED SURGERY ALONG THE TREFOIL KNOT

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Andrew Robert Holmes
B.S., University of Oklahoma, 2012
M.S., Louisiana State University, 2014
May 2017

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Abstract

In this dissertation, we will give a generating set of the Kauffman bracket skein module over the field $\mathbb{Q}(A)$ of 0-framed surgery along the trefoil knot. This generating set is described as a certain subset of a known basis for the skein module over $\mathbb{Z}[A^{\pm 1}]$ of the trefoil exterior.

Chapter 1

Introduction

In [14], Kauffman defined a bracket polynomial on unoriented knot diagrams giving a construction of the Jones polynomial and algebra. The bracket on diagrams of links in S^3 satisfy the relations:

1. $\left\langle \begin{array}{c} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \text{---} \end{array} \right\rangle = A \left\langle \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right\rangle$
2. $\langle \bigcirc \sqcup L \rangle = (-A^2 - A^{-2}) \langle L \rangle$
3. $\langle \emptyset \rangle = 1$

Above, we see a local view of the portions of blackboard framed links inside a 3-ball where the links are the same outside of the 3-ball (represented by the dashed lines).

Przytycki [22] and Turaev [26] (independently) defined the Kauffman bracket skein module of an oriented 3-manifold.

Definition 1.1 (Kauffman bracket skein module). *Let M be a closed orientable 3-manifold, R be a commutative ring with unity and $A \in R$ a unit. The free R -module generated by isotopy classes of unoriented framed links, with the empty link, in M modulo the relations from Kauffman's bracket is the Kauffman bracket skein module of M (denoted $\mathcal{S}(M; R)$).*

One can reinterpret Kauffman's proof that the bracket polynomial is well defined as showing $\mathcal{S}(S^3; \mathbb{Z}[A^{\pm 1}])$ is free on the empty link. The Kauffman bracket is well defined on framed links since up to isotopy the knot diagram can be realized as the core of $S^1 \times [-1, 1]$ (we refer to $[-1, 1]$ as I) with the blackboard framing. The

Kauffman bracket skein module has been a very interesting object of study for many years, with the following manifolds whose Kauffman bracket skein module (or vector space) have been computed over various R :

- Lens spaces [12]
- $S^1 \times S^2$ [13]
- $(2, 2p + 1)$ torus knot exterior [2]
- Certain integral surgeries of the trefoil [3]
- Connect sum of 3-manifolds whose skein modules are known [24]
- Twist knot exterior [4, 6]
- Quaternionic manifold [8, 11]
- Product of disk with two holes and S^1 [21]
- Torus knot exterior [18]
- Certain prism manifolds (with first homology of order 4) [20]
- Two-bridge link exterior [16]
- 3-torus [5, 7]

It is still an open question as to whether every skein module of a closed oriented 3-manifold is finitely generated over $\mathbb{Q}(A)$, the field of rational functions with indeterminate A . With this in mind, we turn our attention to the skein module $\mathcal{S}(\mathcal{M}; \mathbb{Q}(A))$. We let \mathcal{M} denote the closed oriented 3-manifold obtained by 0-framed surgery along the right handed trefoil.

In [3], Bullock studied the skein module (when $R = \mathbb{Z}[A^{\pm 1}]$) of integral surgeries along the right handed trefoil. He proved finite generation in all cases but 0 and 6-framed surgery whereas, these two he showed were infinitely generated. For the skein module (over $\mathbb{Q}(A)$) of 6-framed surgery, finite generation can be shown easily from previous results since it can be realized as $L(2, 1) \# L(3, 1)$ [25, p.271]. In this way, one can see that the dimension of this skein module is four.

Remark 1.2. *This follows from the results in [12] since $\mathcal{S}(L(p, q))$ is a free $\mathbb{Z}[A^{\pm 1}]$ module with $[p/2] + 1$ generators (here $[p/2]$ is the integer part of $p/2$). By [24], the skein module of the connect sum is isomorphic to the tensor product of the skein module of each summand modulo $\mathbb{Z}[A^{\pm 1}]$ torsion. When considered over the field $\mathbb{Q}(A)$, it is exactly the tensor product of the skein modules and thus dimension four.*

Thus, every skein module (over $\mathbb{Q}(A)$) of integer framed surgery with of the trefoil is known to be finite dimensional except 0-framed surgery. We will prove the following theorem regarding $\mathcal{S}(\mathcal{M})$:

Theorem 1.3. *The Kauffman bracket skein module $\mathcal{S}(\mathcal{M}; \mathbb{Q}(A))$ is generated by a subset $\{g_{(x,0,0)} \mid x \geq 0\} \cup \{g_{(1,2,1)}\}$ of a basis for $\mathcal{S}(H_2)$.*

In [2], Bullock proved that the skein module of a trefoil exterior is free on the basis:

$$B = \{m^k \mid k \geq 0\} \cup \{m^k \mathcal{J} \mid k \geq 0\}$$

over $R = \mathbb{Z}[A^{\pm 1}]$. We consider the subset

$$B' = \{m^k \mid k \geq 0\} \cup \{\mathcal{J}\}$$

and prove the following corollary:

Corollary 1.4. *A generating set of $\mathcal{S}(\mathcal{M}; \mathbb{Q}(A))$ is B' .*

In Chapter 2, we define the elements of the basis B , give background on 3-manifolds from surgery, and include results on the skein module of handlebodies and Temperley-Lieb recoupling theory.

In Chapter 3 we prove 1.3 taking an approach similar to Harris [10]. We consider elementary simple closed curves in H_2 and relations given by sliding these curves over attaching curves of 2-handles. Using a Hermitian pairing coming from the doubling of H_2 , we can express elements in $\mathcal{S}(\mathcal{M})$ as linear combinations of an orthogonal basis of $\mathcal{S}(H_2)$, whose coefficients are in $\mathcal{S}(S^3) \cong \mathbb{Q}(A)$ using recoupling theory. Finally, we view the skein module of the trefoil exterior as a module over the skein algebra of a collar neighborhood of its boundary and prove Corollary 1.4.

In Appendices A and B, we provide our diagrammatic evaluations of relations and Mathematica code used to complete our calculations.

Chapter 2

Background

2.1 Surgery descriptions of manifolds

We briefly recall how to produce a 3-manifold from framed surgery along a knot. Let K be a knot in S^3 , and $N(K)$ its regular neighborhood. Recall that $N(K)$ is homeomorphic to $S^1 \times D^2$ (a solid torus). Let J be an essential simple closed curve in $\partial N(K) \sim S^1 \times S^1$. We construct a closed oriented 3-manifold M^3 by $(S^3 \setminus N(K)) \sqcup_h (S^1 \times D^2)$ where h is a homeomorphism mapping a meridional curve from $S^1 \times S^1$ onto J . For further information on surgery descriptions see [9, 25].

In our specific case, let K be the right handed trefoil. We obtain the manifold \mathcal{M} by removing a regular neighborhood of K and sewing in a solid torus specified by the homeomorphism mapping the meridional curve to the longitudinal curve of $S^3 \setminus N(K)$.

Definition 2.1. *A 3-dimensional handlebody of genus g is obtained by attaching g disjoint 1-handles to a 0-handle in such a way to obtain an oriented 3-manifold with boundary Σ_g , a closed surface of genus g .*

For a closed oriented 3-manifold M^3 , we have a handlebody decomposition of into $H_g^1 \sqcup_h H_g^2$ where $H_g^i, i \in \{1, 2\}$ are both handlebodies of genus g and h is a homeomorphism $h : \partial H_g^1 \rightarrow \partial H_g^2$.

Remark 2.2. *We can represent this handlebody decomposition as Heegaard diagram. To do this we consider a Handlebody and mark simple closed curves given by attaching 2-handles to 1-handles. We can represent this as a planar diagram with the attaching curves drawn in the punctured 3-ball. Further information regarding Heegaard diagrams/splittings may be found in [9, 25].*

We will use a genus 2 Heegaard splitting of \mathcal{M} with Heegaard diagram given by Bullock in [3] (Figure 2.1).

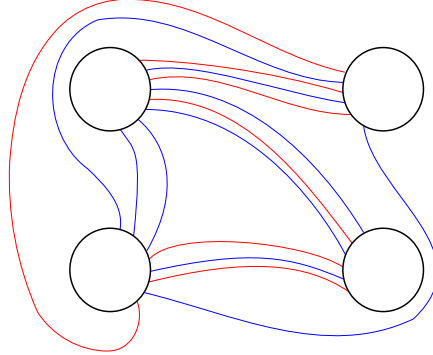


FIGURE 2.1. Heegaard diagram of \mathcal{M} with curves α (the red curve gives Heegaard decomposition of trefoil exterior) and β (the blue curve).

Definition 2.3. For a handlebody H_g , the double of H_g is given by $H_g \sqcup_{id} \bar{H}_g$ where H_g, \bar{H}_g are attached along the identity.

Remark 2.4. We use the standard notation where \bar{H}_g denotes H_g with the orientation reversed.

Theorem 2.5. The double of H_2 is isomorphic to $S^1 \times S^2 \# S^1 \times S^2$.

2.2 A basis for the Kauffman bracket skein module of a trefoil exterior

Bullock in [2], showed that the skein module of trefoil knot (exterior) has a basis which can be described by:

$$B = \{m^k \mid k \geq 0\} \cup \{m^k \mathcal{J} \mid k \geq 0\}.$$

Here m denotes the meridian, \mathcal{J} denotes the curve in the trefoil exterior (Figure 2.2) and we view the skein module of the exterior of the trefoil as a module over the skein algebra of $T^2 \times I$ (where T^2 is homeomorphic to the boundary of the trefoil exterior). This is shown in the notation of m^k, \mathcal{J} , where $m^k \mathcal{J}$ is k -parallel copies of the meridian with \mathcal{J} .

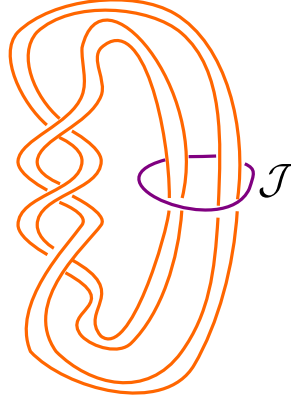


FIGURE 2.2. Exterior of trefoil with \mathcal{J} ($m^0 \mathcal{J}$).

2.3 Banded trivalent graphs

Our notation will follow [15, 19]. For a link, we replace components with colored strands and replace each strand with an idempotent. A strand colored n denotes n parallel copies of a strand colored one. To each strand we can insert an idempotent of the Temperley-Lieb algebra. Using the recursion relation given in Figure 2.3, we can express links as linear combinations of strands without crossing or nullhomologous components.

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} n+1 \\ | \\ \boxed{} \\ | \end{array} & = & \begin{array}{c} n \\ | \\ \boxed{} \\ | \end{array} \\
 \end{array}
 \quad \Bigg| \quad
 \begin{array}{c}
 -\frac{\Delta_n}{\Delta_{n+1}} \begin{array}{c} n-1 \\ | \\ \boxed{} \\ | \end{array} \\
 \end{array}
 \quad \begin{array}{c}
 \begin{array}{c} n \\ | \\ \boxed{} \\ | \end{array} \\
 \begin{array}{c} \boxed{} \\ | \end{array}
 \end{array}
 \end{array}$$

where $\Delta_n = \begin{array}{c} n \\ | \\ \boxed{} \\ | \end{array}$

FIGURE 2.3. Recursion relation for Jones-Wenzl idempotents.

Remark 2.6. In Figure 2.3, Δ_n is the n -th Chebyshev polynomial with $\Delta_n = (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}$. Computations are carried out by evaluating quantum integers $[n+1] = (-1)^{n-1} \Delta_n$.

Definition 2.7. A triple (x, y, z) of non-negative integers (Figure 2.4) is admissible if $x + y + z \equiv 0 \pmod{2}$ and $|x - z| \leq y \leq x + z$ (triangle inequality).

We define a trivalent vertex given an admissible triple (x, y, z) as in Figure 2.4. One can expand the right hand side of the figure, using the recursion relations of the idempotents and see that these vertices are linear combinations of links in $\mathcal{S}(S^1 \times I)$. The numbers $i = (a + b - c)/2, j = (a + c - b)/2, k = (b + c - a)/2$ are known as the internal colors and a vertex is admissible if and only if internal colors can be found for it.

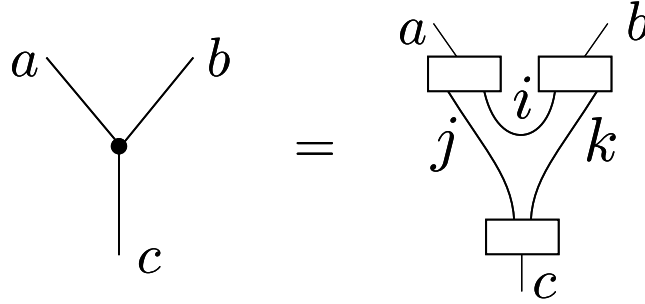


FIGURE 2.4. Trivalent vertex (a, b, c) defined.

In [19], Masbaum and Vogel give an algorithm to evaluate these graphs as scalar multiples of the empty link in $\mathcal{S}(S^3)$. We give the following formulas (as in [15]) and theorems (as in [1, 8, 10]) which allow us to calculate the value in $\mathcal{S}(D^3; \mathbb{Q}(A)) \cong \mathbb{Q}(A)$ of banded trivalent graphs in S^3 . For further discussion on banded trivalent graphs see [15, 19].

$$\begin{array}{c} a \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} b \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} c \\ \text{---} \end{array} = \theta(a, b, c) \quad (2.1)$$

$$\begin{array}{c} B \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} C \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} E \\ \text{---} \end{array} = Tet \begin{bmatrix} A & B & E \\ C & D & F \end{bmatrix} \quad (2.2)$$

$$\begin{array}{c} \text{Diagram: A triangle with vertices labeled } a, b, c \text{ and edges labeled } e, f. \end{array} = \begin{cases} \frac{Tet \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}}{\theta(a,d,e)} \begin{array}{c} \text{Diagram: A vertex with three edges labeled } a, d, e. \end{array}, & \text{if } (a,d,e) \text{ is admissible} \\ 0, & \text{otherwise} \end{cases} \quad (2.3)$$

$$\begin{array}{c} \text{Diagram: A loop with vertices labeled } a, b, c. \end{array} = \lambda_c^{ab} \begin{array}{c} \text{Diagram: A vertex with three edges labeled } a, b, c. \end{array} \quad (2.4)$$

$$\begin{array}{c} \text{Diagram: A loop with vertices labeled } a, b, c, d. \end{array} = \frac{\delta_d^a \theta(a, b, c)}{\Delta_a} \begin{array}{c} \text{Diagram: A vertex with two edges labeled } a, a. \end{array} \quad (2.5)$$

$$\begin{array}{c} \text{Diagram: A vertex with four edges labeled } a, b, c, d. \end{array} = \sum_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} \begin{array}{c} \text{Diagram: A vertex with four edges labeled } a, b, c, d. \end{array} \quad (2.6)$$

where

$$\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} = \frac{Tet \begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix} \Delta_i}{\theta(a, d, i) \theta(b, c, i)}.$$

Theorem 2.8. *Fusion Formula*

$$\begin{array}{c} \text{Diagram: A vertex with two edges labeled } a, b. \end{array} = \sum_i \frac{\Delta_i}{\theta(a, b, i)} \begin{array}{c} \text{Diagram: A vertex with four edges labeled } a, b, a, b. \end{array}$$

where the sum is over all i such that (a, b, i) is admissible.

Theorem 2.9. *2-Sphere Reduction*

If a sphere intersects a skein element in exactly 2 labelled arcs, then:

$$\begin{array}{c} a \quad b \\ \vdots \quad \vdots \\ \text{---} \text{---} \text{---} \\ \vdots \quad \vdots \end{array} = \frac{\delta_b^a}{\Delta_a} \begin{array}{c} a \\ \cup \\ \text{---} \text{---} \text{---} \\ \cap \\ a \end{array}$$

If a sphere intersects a skein element in exactly 3 labelled arcs, then:

$$\begin{array}{c} a \quad b \quad c \\ \vdots \quad \vdots \quad \vdots \\ \text{---} \text{---} \text{---} \\ \vdots \quad \vdots \quad \vdots \end{array} = \begin{cases} \frac{1}{\theta(a,b,c)} \begin{array}{c} a \quad b \quad c \\ \cup \quad \cup \quad \cup \\ \text{---} \text{---} \text{---} \\ \cap \quad \cap \quad \cap \\ a \quad b \quad c \end{array}, & \text{if } (a,b,c) \text{ is admissible} \\ 0, & \text{otherwise} \end{cases}$$

If a sphere intersects a skein element in $n > 3$ labelled arcs, then:

$$\begin{array}{c} a_1 \quad a_2 \quad \dots \quad a_{n-1} \quad a_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array} = \sum_{\mathcal{I}} \frac{1}{\begin{array}{c} c_1 \quad \dots \quad c_{n-3} \\ \cup \quad \cup \quad \cup \\ \text{---} \text{---} \text{---} \\ \cap \quad \cap \quad \cap \\ c_1 \quad \dots \quad c_{n-3} \end{array}} \begin{array}{c} a_1 \quad a_2 \quad \dots \quad a_{n-1} \quad a_n \\ \cup \quad \cup \quad \cup \quad \cup \quad \cup \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \cap \quad \cap \quad \cap \quad \cap \quad \cap \\ a_1 \quad a_2 \quad \dots \quad a_{n-1} \quad a_n \end{array}$$

where \mathcal{I} consists of all admissible colorings of c_1, \dots, c_{n-3} .

Remark 2.10. We can work out using 2.5, and 2.1 that

$$\begin{array}{c} c_1 \quad \dots \quad c_{n-3} \\ \cup \quad \cup \quad \cup \\ \text{---} \text{---} \text{---} \\ \cap \quad \cap \quad \cap \\ c_1 \quad \dots \quad c_{n-3} \end{array} a_n = \frac{\theta(a_1, a_2, c_1) \theta(c_{n-3}, a_{n-1}, a_n) \prod_{i=1}^{n-3} \theta(c_i, c_{i+1}, a_{i+2})}{\prod_{j=1}^{n-3} \Delta_{c_j}}$$

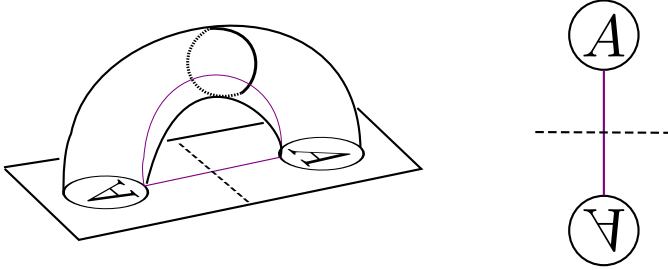
For our proof it will also be helpful to consider idempotent elements in the skein module of the solid torus $D^2 \times S^1$. We give the following Lemma [17, p.156] [19, p.366]

Lemma 2.11 (Multiplication in $\mathcal{S}(D^2 \times S^1)$). *In $\mathcal{S}(D^2 \times S^1)$, we have*

$$\begin{array}{c} \text{circle} \\ \text{red arc } a \\ \text{blue line } b \\ \text{circle} \end{array} = \sum_c \begin{array}{c} \text{circle} \\ \text{purple line } c \\ \text{circle} \end{array}$$

where the sum is over all c , such that (a, b, c) is admissible.

Remark 2.12. *In the diagram above, we use the convention where a vertical pair of circles denote where the 1-handles are attached to the 3-ball. The circles are identified by reflection about the horizontal axis in the page.*



2.4 Graph-basis of $\mathcal{S}(H_2)$

Definition 2.13. *A colored trivalent graph in 3-manifold is said to be admissible if the three colors of the edges meeting at any vertex are admissible.*

Thus, a trivalent graph $g_{(x,y,z)}$ (Figure 2.5) is said to be admissible if the vertices (x, x, y) and (y, z, z) are admissible.

Remark 2.14. *Using Definition 2.7 we see $x, y, z \geq 0$ must satisfy:*

$$0 \leq y \leq 2x,$$

$$0 \leq y \leq 2z,$$

$$y \equiv 0 \pmod{2}.$$

As in [8, 10], a basis for $\mathcal{S}(H_2)$ is given by all admissible trivalent graphs $g_{(a,b,c)}$. We denote this graph-basis of $\mathcal{S}(H_2)$ as $\mathcal{B} = \{g_{(a,b,c)}\}$. Given the genus 2 Heegaard splitting of \mathcal{M} , $\iota_* : \mathcal{S}(H_2) \hookrightarrow \mathcal{S}(\mathcal{M})$ induced by inclusion is surjective as noted by

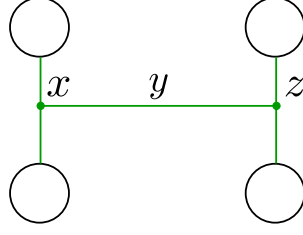


FIGURE 2.5. The graph $g_{(x,y,z)}$ in $\mathcal{S}(H_2)$.

Przytycki [23, 2.2]. Relations that we will describe in Chapter 3 will be given in terms of graph-basis elements of $\mathcal{B} \subset \mathcal{S}(H_2)$.

2.4.1 A linear ordering on \mathcal{B}

Suppose that $g_{(x,y,z)}, g_{(x',y',z')} \in \mathcal{B}$ and $s = x + z, s' = x' + z'$. We have $g_{(x,y,z)} > g_{(x',y',z')}$ if

$$\begin{aligned} s &> s' \text{ or} \\ s &= s', z > z' \text{ or} \\ s &= s', z = z', \text{ and } y > y' \end{aligned} \tag{2.7}$$

It is worth noting when $s = s'$ and $z = z'$ then necessarily $x = x'$.

Definition 2.15. $g_{(x,y,z)} \in \mathcal{S}(H_2)$ is said to be rewritten if

$$g_{(x,y,z)} - \sum_j \eta_j g_{(a_j,b_j,c_j)} \in \ker \iota_*$$

where $\eta_j \in \mathbb{Q}(A)$ and $g_{(a_j,b_j,c_j)} < g_{(x,y,z)}$.

Remark 2.16. For $g_{(x,y,z)}, g_{(p,q,r)} \in \mathcal{B}$ we will use the notation $g_{(x,y,z)} \sim g_{(p,q,r)}$ if

$$g_{(x,y,z)} - g_{(p,q,r)} \in \ker \iota_*.$$

2.5 A Hermitian pairing on $\mathcal{S}(H_2) \times \mathcal{S}(H_2)$

We can relate skeins in $\mathcal{S}(H_2)$ by sliding over attaching curves of 2-handles on H_2 .

These skeins can be written as $\sum_i \eta_i g_{(a_i,b_i,c_i)}$ with graph-basis elements $g_{(a_i,b_i,c_i)}$ and coefficients $\eta_i \in \mathbb{Q}(A)$. The coefficients can be calculated by considering a Hermitian pairing defined on the double of H_2 :

$$\langle \cdot, \cdot \rangle : \mathcal{S}(H_2) \times \mathcal{S}(H_2) \rightarrow \mathbb{Q}(A)$$

this pairing is described further in [8]. We can use results from recoupling theory and orthogonality of our graph-basis with this pairing to write a skein $g_{(x,y,z)} \in \mathcal{S}(H_2)$ as,

$$g_{(x,y,z)} = \sum_i \eta_i g_{(a_i,b_i,c_i)} \quad (2.8)$$

where $\eta_i = \frac{\langle g_{(x,y,z)}, g_{(a_i,b_i,c_i)} \rangle}{\langle g_{(a_i,b_i,c_i)}, g_{(a_i,b_i,c_i)} \rangle}$.

In Figure 2.6, we show the pairing $\langle g_{(a,b,c)}, g_{(a,b,c)} \rangle$, which is the denominator of η in 2.8. Even though the pairing occurs in the double of H_2 , we can push $g_{(a,b,c)}$ into the innermost H_2 and consider the picture of the pairing in the 3-ball. This is how we will show the pairing diagrammatically.

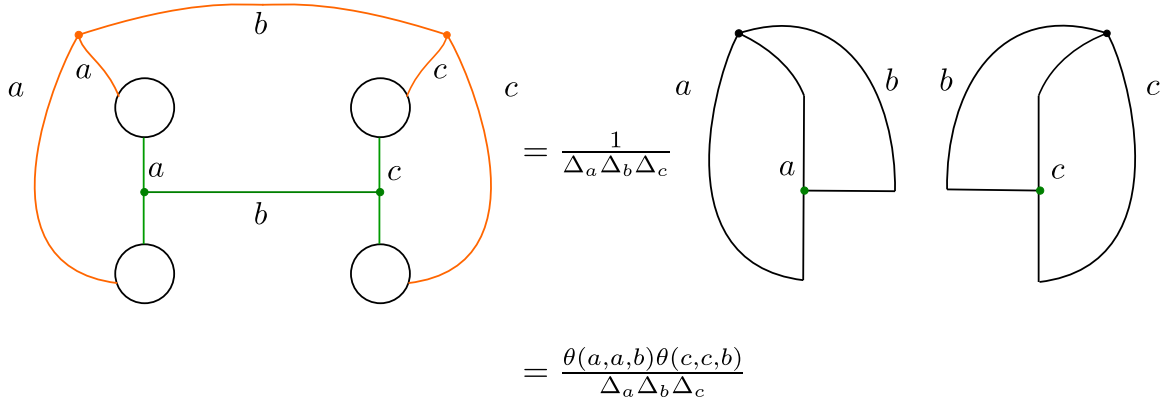


FIGURE 2.6. $\langle g_{(a,b,c)}, g_{(a,b,c)} \rangle$ realized as a scalar multiple after three 2-sphere reductions in $S^1 \times S^2 \# S^1 \times S^2$.

Chapter 3

A generating set of $\mathcal{S}(\mathcal{M})$

We will show that $\mathcal{S}(\mathcal{M})$ is generated by a subset of our graph-basis $\mathcal{B} \subset \mathcal{S}(H_2)$. We will rewrite skeins $g_{(x,y,z)} \in \mathcal{S}(H_2)$ modulo relations given by sliding essential simple closed curves in $\mathcal{S}(H_2)$ over attaching curves of 2-handles specifying \mathcal{M} . For the relations we use, we provide diagrams showing explicitly how the relations are obtained and explain notation. In Appendix A, we detail the evaluation of pairing relations with graph basis elements diagrammatically (as shown in Figure 2.6) with Mathematica code implementing the evaluation in Appendix B.

3.1 Relation A_1 and some consequences

We consider the relation given by sliding a skein in H_2 over the attaching curve of a 2-handle with attaching curve α . The skein we consider is the union of a graph-basis element and an essential simple closed curve, where only the latter is slid over the attaching curve. In the usual manner, this slide is realized by band summing the skein with the attaching curve α . This specific relation is given in Figure 3.1 and is denoted $A(\omega, \alpha, \beta, \gamma)$. We will denote relations given by sliding skeins in H_2 over the attaching curve α (β) as A_\square (B_\square) respectively.

Remark 3.1. When $\omega = 1$, we will use the shorthand $A_1(\alpha, \beta, \gamma)$.

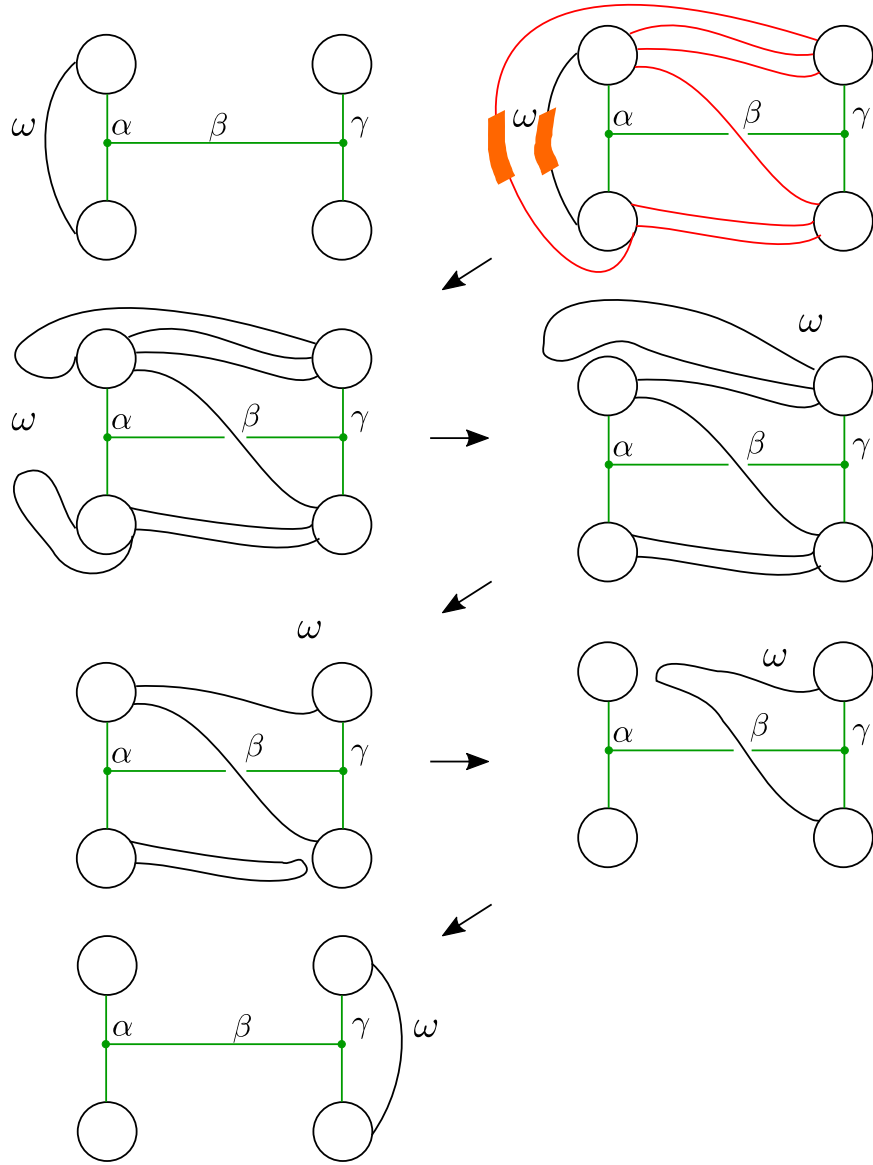


FIGURE 3.1. The relation $A(\omega, \alpha, \beta, \gamma)$ as a result of band sum with curve α (the orange regions denote where the band sum occurs).

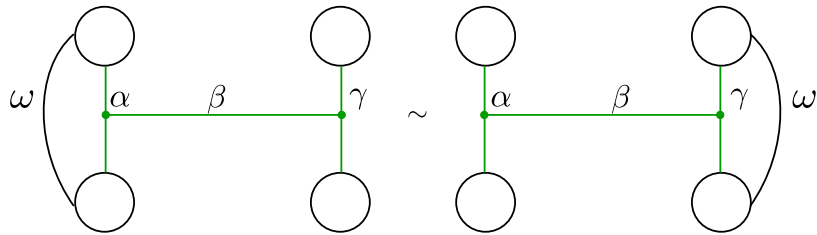


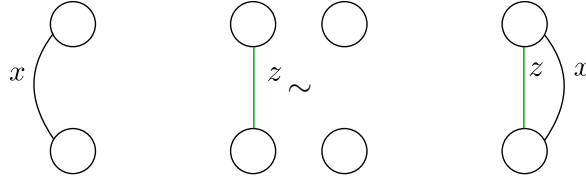
FIGURE 3.2. The relation $A(\omega, \alpha, \beta, \gamma)$.

We will use relation $A(\omega, \alpha, \beta, \gamma)$ (Figure 3.2) to rewrite $g_{(x,0,z)}$ when $z \neq 0$. Our proof splits into two cases:

- $x < z$
- $x > z$ and $z > 0$.

Lemma 3.2. $g_{(x,0,z)} \sim g_{(z,0,x)}$.

Proof. In Figure 3.3, $g_{(x,0,z)}$ is shown on the left hand side. Using relation $A(x, 0, 0, z)$,



we can think of these as two idempotents colored z, x in $\mathcal{S}(D^2 \times S^1)$ and perform multiplication of the closures of the idempotents as in Lemma 2.11 to obtain the left hand equality (Figure 3.3). Similarly, we can obtain the right hand inequality from $A(z, 0, 0, x)$ using the same argument. \square

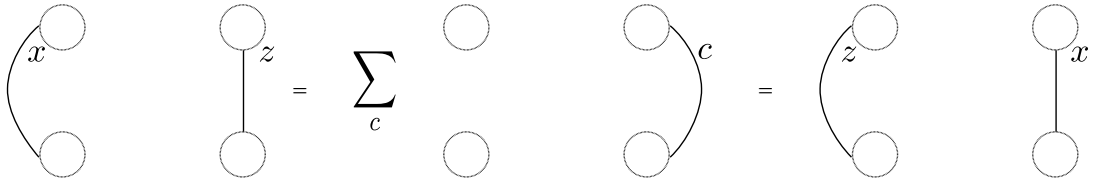
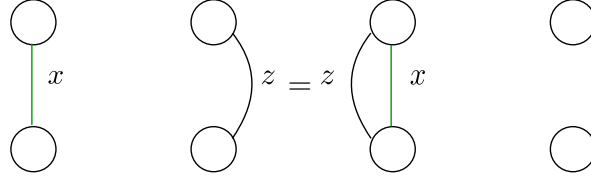


FIGURE 3.3. $g_{(x,0,z)} \sim g_{(z,0,x)}$ when $x < z$ and $z > 0$.

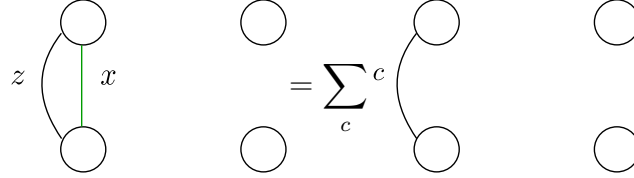
Rather than looking at the left hand side of relation $A(z, x, 0, 0)$ (denoted $L(A(z, x, 0, 0))$), we look at the right hand side (denoted $R(A(z, x, 0, 0))$) and move the z colored strand from the right to left hand side.

Lemma 3.3. $g_{(x,0,z)} \sim \sum_c g_{(c,0,0)}$ for all c such that (x, z, c) is admissible.

Proof. $R(A(z, x, 0, 0))$ is equal to $g_{(x,0,z)}$.



As in Lemma 3.2, these strands are in the right hand 1-handle. By Lemma 2.11



where the sum is over all c such that (x, z, c) is admissible. By admissibility, $g_{(x,0,z)} \sim g_{(x+z,0,0)} + \sum_j \tilde{g}_j$ (where $\tilde{g}_j \in \mathcal{B}$ are lower in our ordering than $g_{(x,0,z)}$) with $g_{(x,0,z)} > g_{(x+z,0,0)}$. \square

We are now ready to prove the following lemma:

Lemma 3.4. $g_{(x,0,z)}$ can be rewritten for $z \neq 0$.

Proof. In the case when $x < z$, by Lemma 3.2 $g_{(x,0,z)} \sim g_{(z,0,x)}$ which is lower in our ordering. When $0 < z < x$, by Lemma 3.3 $g_{(x,0,z)} \sim g_{(x+z,0,0)} +$ lower order terms. \square

The graph-basis elements which are not rewritten by Lemma 3.4, are $g_{(x,y,z)}$ for $y, z \neq 0$ and $g_{(x,0,0)}$.

3.2 Writing relation A_1 using orthogonality

The relation $A_1(\alpha, \beta, \gamma)$ can be given in terms of our graph-basis (as in 2.8) by viewing the left (denoted $L(A_1(\alpha, \beta, \gamma))$) and right hand side (denoted $R(A_1(\alpha, \beta, \gamma))$) in the double of H_2 . Figure 3.4 shows $\langle L(A_1(\alpha, \beta, \gamma)), g_{(a,b,c)} \rangle$ in H_2 .

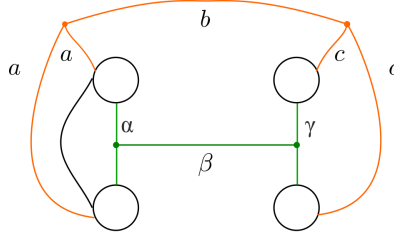


FIGURE 3.4. The pairing of $L(A_1(\alpha, \beta, \gamma))$ with graph-basis element $g_{(a,b,c)}$ in the 3-ball of H_2 .

Below we have $L(A_1(\alpha, \beta, \gamma))$ written in this way:

$$\sum_{(a,b,c)} \frac{\left\langle \begin{array}{c} \text{Graph 1} \\ \text{Graph 2} \end{array}, \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array} \right\rangle}{\left\langle \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array}, \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array} \right\rangle} \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array},$$

which we can evaluate as:

$$\begin{aligned} & \sum_{(a,b,c)} \frac{\left\langle \begin{array}{c} \text{Graph 1} \\ \text{Graph 2} \end{array}, \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array} \right\rangle}{\left\langle \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array}, \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array} \right\rangle} \begin{array}{c} \text{Graph 3} \\ \text{Graph 4} \end{array} \\ &= \sum_{(a,b,c)} \frac{\langle L(A_1(\alpha, \beta, \gamma)), g_{(a,b,c)} \rangle}{\langle g_{(a,b,c)}, g_{(a,b,c)} \rangle} g_{(a,b,c)} \end{aligned}$$

where the sum is over all (a, b, c) admissible.

Remark 3.5. We will extend the notation $L(REL(\alpha, \beta, \gamma))$ (similarly $R(REL(\alpha, \beta, \gamma))$) to denote the left hand side (right hand side) of relation REL . When we refer to the coefficient of relation REL , we mean $\frac{\langle L(REL(\alpha, \beta, \gamma)) - R(REL(\alpha, \beta, \gamma)), g_{(a,b,c)} \rangle}{\langle g_{(a,b,c)}, g_{(a,b,c)} \rangle}$. In the code in Appendix B, this coefficient is denoted $REL[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}, \mathbf{c}]$ where $(\alpha, \beta, \gamma) = (x, y, z)$.

In order to rewrite $g_{(1,2,z)}$, we will use admissibility and orthogonality of our basis $g_{(a,b,c)}$ with respect to the pairing of relation $A_1(1, 2, z - 1)$.

Lemma 3.6. $g_{(1,2,z)}$ can be rewritten for $z \geq 2$.

Proof. By orthogonality and admissibility, we have $g_{(a,b,c)}$ appears in the pairing of $A_1(1, 2, z - 1)$ only if:

$$(a, b, c) = \begin{cases} (1, 2, z) \\ (2, 2, z - 1) \\ (1, 2, z - 2) \end{cases}$$

Notice, $g_{(1,2,z)}$ is the highest element which appears and we have:

$$g_{(1,2,z)} = g_{(2,2,z-1)} + \frac{A^{4z} + A^{4(z+3)} - A^{8z+4} - A^8}{A^4(-A^{4z} + A^{8z} - A^{4z+4} + A^4)} g_{(1,2,z-2)}$$

where $g_{(1,2,z)} > g_{(2,2,z-1)} > g_{(1,2,z-2)}$ in our ordering for $z \geq 3$. When $z = 2$, $g_{(1,2,2)} = g_{(2,2,1)}$. \square

Lemma 3.6 rewrites all graph-basis elements of the form $g_{(1,2,z)}$ except $g_{(1,2,1)}$. We now turn our attention to rewriting elements of the form $g_{(x,y,z)}$ when $y \geq 2$ and z is non-zero.

3.3 Relation B_4

We now consider a relation given by sliding a skein in H_2 over the attaching curve β of a 2-handle (Figure 3.6). The relation is detailed in Figure 3.5.

Lemma 3.7. $g_{(x,y,z)}$ can be rewritten for $2 \leq y \leq 2x - 2, x \geq 2$, and $z \geq 1$.

Proof. Using $B_4(x - 2, y - 2, z - 1)$ for $2 \leq y \leq 2x - 2, x \geq 2$, and $z \geq 1$, we have

$$2 \leq y \leq 2x - 2,$$

$$2 \leq y \leq 2z,$$

$$y \equiv 0 \pmod{2}.$$

due to admissibility from Definition 2.13. The highest term appearing in this relation is $g_{(x,y,z)}$ with coefficient $B_4(x - 2, y - 2, z - 1)$ equal to $-A^{-2x+2y}$ which is non-zero. \square

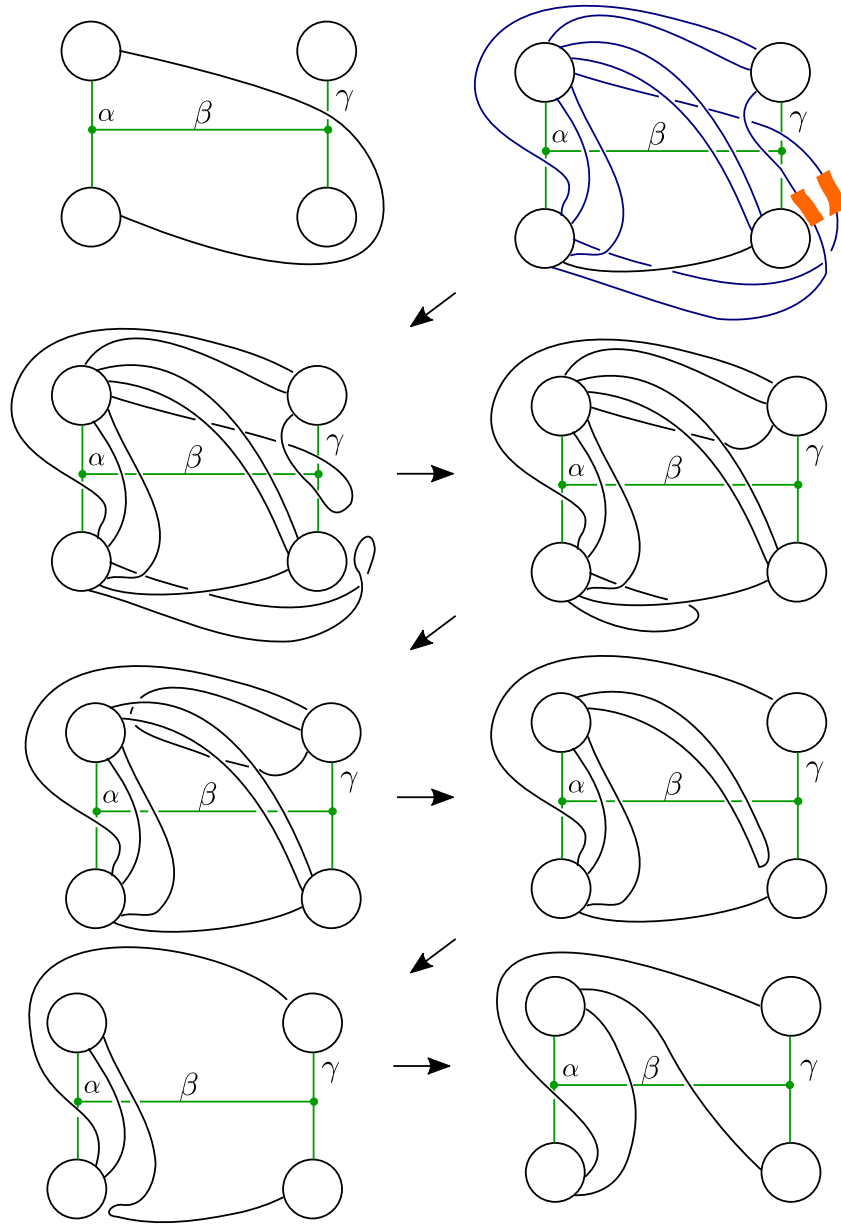


FIGURE 3.5. The relation $B_4(\alpha, \beta, \gamma)$ as a result of band sum with curve β .

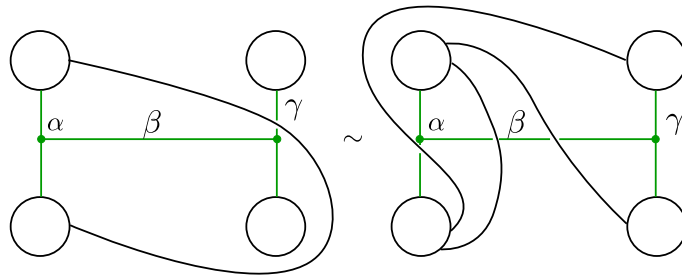


FIGURE 3.6. The relation $B_4(\alpha, \beta, \gamma)$.

Using Lemma 3.7, we have rewritten all graph-basis elements except those of the form $g_{(1,2,1)}$, $g_{(1,0,1)}$, $g_{(1,0,0)}$, $g_{(0,0,1)}$ and $g_{(x,2x,z)}$. After using Lemma 3.3, the elements which remain to be rewritten are of the form $g_{(1,2,1)}$, $g_{(x,0,0)}$ and $g_{(x,2x,z)}$.

3.4 Relation A_2

We will use a relation given by sliding a skein in H_2 over the attaching curve α of a 2-handle (Figures 3.8, 3.7).

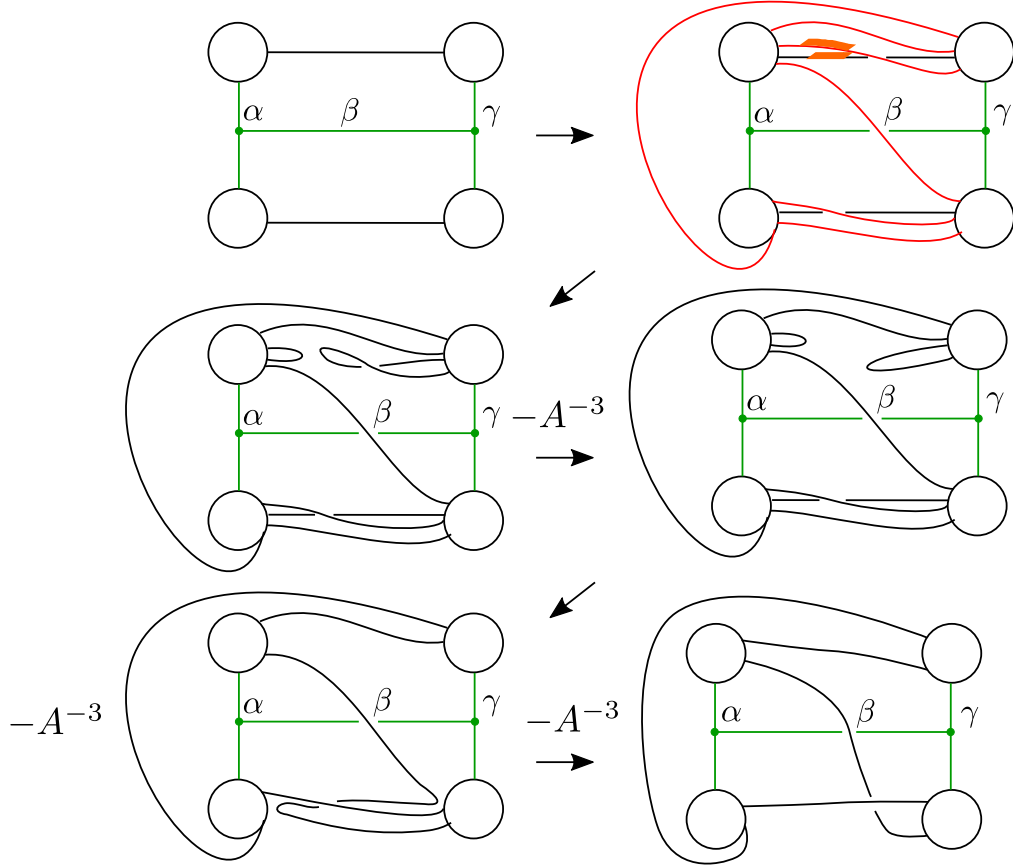


FIGURE 3.7. The relation $A_2(\alpha, \beta, \gamma)$ as a result of band sum with curve α .

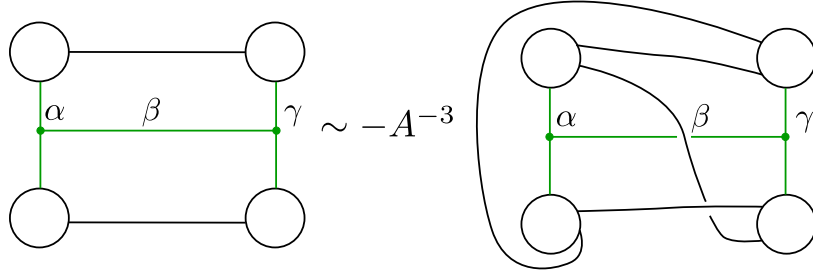


FIGURE 3.8. The relation A_2 .

Lemma 3.8. $g_{(x,2x,z)}$ can be rewritten for $x, z \geq 2$.

Proof. Using relation $A_2(x-2, 2x-4, z-2)$ with $x, z \geq 2$ we have the highest term appearing is $g_{(x,2x,z)}$ with coefficient $A_2(x-2, 2x-4, z-2)$ equal to $-A^{4x-6}$ which is non-zero. \square

Remark 3.9. Though we specifically look at $y = 2x$, the code in Appendix 3.9 for which the calculation is carried out, is evaluated first for y even and then $y = 2x$.

Since $x, z \geq 2$, Lemma 3.8 does not rewrite $g_{(1,2,1)}$.

3.5 Main result and corollary

We now recall Theorem 1.3.

Theorem 1.3. The Kauffman bracket skein module $\mathcal{S}(\mathcal{M}; \mathbb{Q}(A))$ is generated by a subset $\{g_{(x,0,0)} \mid x \geq 0\} \cup \{g_{(1,2,1)}\}$ of a basis for $\mathcal{S}(H_2)$.

Proof. From the previous lemmas, we can rewrite all graph-basis elements except $g_{(x,0,0)}$ for $x \geq 0$ and $g_{(1,2,1)}$, which span $\mathcal{S}(\mathcal{M})$. \square

As we have noted in Chapter 2, Bullock showed the skein module of the trefoil exterior to be free on the basis:

$$B = \{m^k \mid k \geq 0\} \cup \{m^k \mathcal{J} \mid k \geq 0\}.$$

We will consider the subset of B :

$$B' = \{m^k \mid k \geq 0\} \cup \{\mathcal{J}\}$$

and recall Corollary 1.4.

Corollary 1.4. *A generating set of $\mathcal{S}(\mathcal{M}; \mathbb{Q}(A))$ is B' .*

Proof. We have shown that we can rewrite skeins in $\mathcal{S}(H_2)$ as linear combinations of our graph-basis except for elements $g_{(x,0,0)}$, $g_{(1,2,1)}$. Since m^k is k -parallel copies of the meridian which is the basis element $g_{(0,0,k)}$ where the strands are idempotents. It follows from relation A_1 that $g_{(0,0,k)} \sim g_{(k,0,0)}$. Next, we can realize $g_{(1,2,1)}$ as a linear combination of the following skeins by expanding the idempotent according to the recursion relation in Figure 2.3 to get:

The diagram shows a skein relation for $g_{(1,2,1)}$. On the left, a diagram with four circles (two on the left, two on the right) connected by a central horizontal line with a '2' above it. This is equal to a diagram with two horizontal lines (top and bottom) connecting the left and right circles, labeled $h(\mathcal{J})$ below, minus $\frac{1}{\Delta_1}$ times a diagram with two vertical lines connecting the top and bottom circles, labeled $g_{(1,0,1)}$ below. Below the diagrams is the equation:

$$g_{(1,2,1)} = h - \frac{1}{\Delta_1} g_{(1,0,1)}.$$

Where h is shown as the skein below. The element h can be realized as \mathcal{J} in the trefoil exterior. Using Lemma 3.4, $g_{(1,0,1)} \sim g_{(2,0,0)} + g_{(0,0,0)}$. This means that $g_{(1,2,1)}$ is a linear combination of \mathcal{J} and $m^2 \in B'$.

Remark 3.10. *We will show that the element h is easily realized as \mathcal{J} by considering the handle decomposition of the trefoil exterior (recall the curve α specifies this). We have the decomposition into one 0-handle, two 1-handles, one 2-handle and one 3-handle. This is shown in the upper left hand corner of the following diagram. To this handle decomposition, we add a 2-handle (specified by the black line) to cancel the right most 1-handle. The co-core of the 2-handle is isotopic in the 0-handle to the proper arc colored orange with two endpoints on the boundary of the 3-ball. The arc is proper if the embedding of the arc joins the boundary of the arc with the boundary of the 3-ball.*

We now show that the skein h is actually $\mathcal{J} \in B'$ which follows from Figure 3.9.

Starting in the upper left hand corner (and following the steps below):

- 1: Perform a handle cancellation in the usual way by attaching the 2-handle to the right 1-handle.
- 2-5: Isotope the α curve along ∂H_1 careful to not let the curve pass over the marked points on the orange curve. The orange arc has two points on the boundary where the arc is in the 3-ball. We also isotope this curve along the boundary of the 3-ball.
- 6: Isotope the α curve along the boundary of the attaching disks of the 1-handle.
- 7: Isotopy.
- 8: Cancel the 1-handle with a 2-handle. Now we are left with a 3-ball with boundary.
- 9: Isotopy.
- 10: Add in the 3-handle along the boundary resulting in S^3 .
- 11-13: Isotopy in S^3 , yielding $\mathcal{J} \in B'$.

□

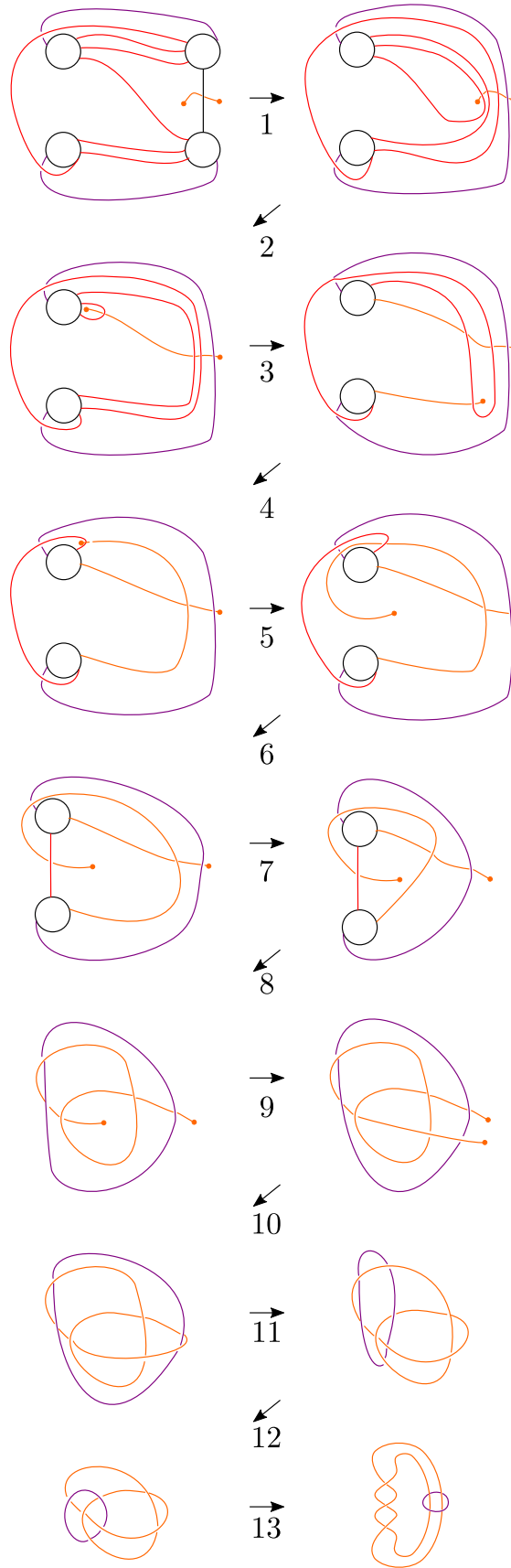


FIGURE 3.9. Realizing a skein in $\mathcal{S}(H_2)$ as \mathcal{J} .

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Appendix A: Relations in $\mathcal{S}(\mathcal{M})$ and Diagrammatic Evaluations

Our notation of relations and common trivalent graphs will follow Chapter 2. Evaluations of common trivalent graphs in terms of quantum factorials will follow [15].

With $LHSRel$ (similarly $RHSRel$) being viewed as trivalent graphs, we use our re-coupling formulas with 2-sphere reduction to evaluate:

- $LHSRelF$ denotes the factors coming from three 2-sphere reductions in the connected sum $S^1 \times S^2 \# S^1 \times S^2$
- $LHSRelL$ denotes the left side of the diagram after 2-sphere reductions
- $LHSRelR$ denotes the right side of the diagram after 2-sphere reductions
- $LHSRelTerm$ ($RHSRelTerm$ respectively) denotes the product $(LHSRelF)(LHSRelL)(LHSRelR)$
- $LHSRel$ denotes the sum of $LHSRelTerm$ over admissible colorings after the three 2-sphere reductions.

Remark 3.11. *We will also use*

$$Tri \begin{bmatrix} A & B & E \\ C & D & F \end{bmatrix}$$

to denote

$$\frac{Tet \begin{bmatrix} A & B & E \\ C & D & F \end{bmatrix}}{\theta(A, D, E)}.$$

This is used to denote the collapsing of a triangular face as shown in 2.3. Below, before we perform such a collapse, we color the triangle pink.

3.6 Relation $A(1, \alpha, \beta, \gamma)$

In this relation, the right hand side is the left hand side rotated by π . A rotation by π on H_2 induces a map on $\mathcal{S}(H_2)$ where $g(x, y, z) \mapsto g(z, y, x)$.

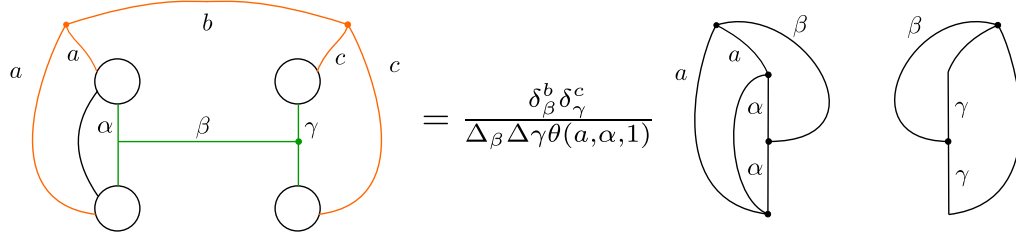


FIGURE 3.10. The left hand side of A_1 ($LHSA_1$).

$$\text{Diagram} = Tet \begin{bmatrix} \alpha & a & \beta \\ a & \alpha & 1 \end{bmatrix}$$

FIGURE 3.11. Detail of $LHSA_1L$.

$$\text{Diagram} = \theta(\gamma, \gamma, \beta)$$

FIGURE 3.12. Detail of $LHSA_1R$.

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(* Relation A1*)

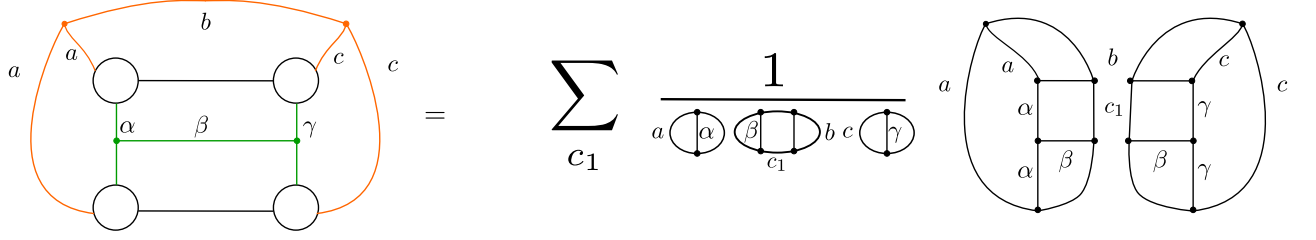
lhsA1[x_, y_, z_, a_, b_, c_] := If[adm[a, x, 1] && c == z && b == y,
  theta[z, z, y] tet[x, a, a, x, y, 1] / delta[y] / delta[z] / theta[a, x, 1],
  0];
rhsA1[x_, y_, z_, a_, b_, c_] := lhsA1[z, y, x, c, b, a];
A1[x_, y_, z_, a_, b_, c_] :=
  If[adm[x, x, y] && adm[z, z, y] && adm[a, a, b] && adm[c, c, b],
    (lhsA1[x, y, z, a, b, c] - rhsA1[x, y, z, a, b, c]) / norm[a, b, c], 0] // Simplify;

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FIGURE 3.13. A_1 Mathematica code.

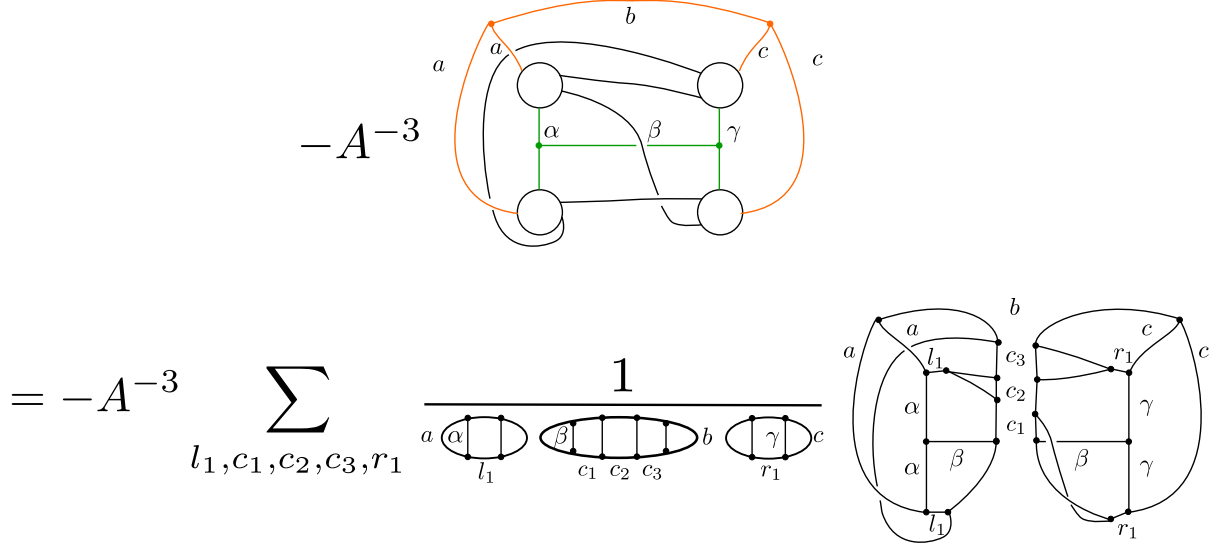
3.7 Relation A_2

This is relation $A_2(\alpha, \beta, \gamma)$.



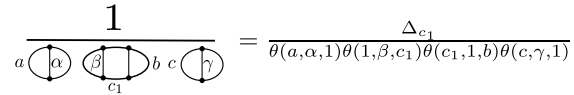
$$\text{Diagram} = \sum_{c_1} \frac{1}{\text{Diagram}}$$

FIGURE 3.14. The left hand side of A_2 ($LHSA_2$).



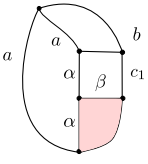
$$-A^{-3} \text{Diagram} = -A^{-3} \sum_{l_1, c_1, c_2, c_3, r_1} \frac{1}{\text{Diagram}}$$

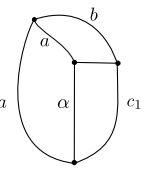
FIGURE 3.15. The right hand side of A_2 ($RHSA_2$).



$$\frac{1}{\text{Diagram}} = \frac{\Delta_{c_1}}{\theta(a, \alpha, 1) \theta(1, \beta, c_1) \theta(c_1, 1, b) \theta(c, \gamma, 1)}$$

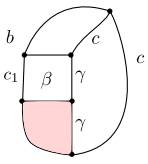
FIGURE 3.16. Detail of $LHSA_2F$.

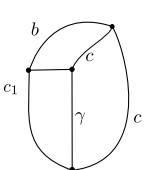


$$= g_1 \quad \text{where } g_1 = \text{Tri} \begin{bmatrix} a & \alpha & \alpha \\ \beta & c_1 & 1 \end{bmatrix}$$


$$= g_1 \text{ Tet} \begin{bmatrix} a & b & c_1 \\ 1 & \alpha & a \end{bmatrix}$$

FIGURE 3.17. Detail of $LHSA_2L$.



$$= h_1 \quad \text{where } h_1 = \text{Tri} \begin{bmatrix} \gamma & \gamma & c \\ 1 & c_1 & \beta \end{bmatrix}$$


$$= h_1 \text{ Tet} \begin{bmatrix} c_1 & b & c \\ c & \gamma & 1 \end{bmatrix}$$

FIGURE 3.18. Detail of $LHSA_2R$.

$$\frac{1}{\overbrace{a \quad \alpha \quad l_1} \quad \overbrace{c_1 \quad c_2 \quad c_3} \quad \overbrace{b \quad \gamma \quad r_1} \quad c} = \frac{\Delta_{l_1} \Delta_{c_1} \Delta_{c_2} \Delta_{c_3} \Delta_{r_1}}{\theta(a, \alpha, l_1) \theta(l_1, 1, 1) \theta(1, \beta, c_1) \theta(c_1, c_2, 1) \theta(c_2, c_3, 1) \theta(c_3, 1, b) \theta(1, 1, r_1) \theta(r_1, \gamma, c)}$$

FIGURE 3.19. Detail of $RHSA_2F$.

\begin{aligned}
&= g_1 \sim \text{where } g_1 = \text{Tri} \begin{bmatrix} l_1 & 1 & c_3 \\ c_2 & c_1 & 1 \end{bmatrix} \\
&= g_2 \sim \text{where } g_2 = \lambda_{c_3}^{b_1} \\
&= g_{3,i} \text{ where } g_{3,i} = g_2 \sum_i \begin{Bmatrix} a & \alpha & i \\ 1 & 1 & l_1 \end{Bmatrix} \\
&= g_{4,i} \text{ where } g_{4,i} = g_{3,i} \text{Tri} \begin{bmatrix} \alpha & \beta & c_1 \\ 1 & i & \alpha \end{bmatrix} \\
&= g_{5,i} \text{ where } g_{5,i} = g_{4,i} \text{Tri} \begin{bmatrix} a & l_1 & c_3 \\ c_1 & i & \alpha \end{bmatrix} \\
&= g_{5,i} \text{Tet} \begin{bmatrix} 1 & a & b \\ a & c_3 & i \end{bmatrix}
\end{aligned}

FIGURE 3.20. Detail of $RHSA_2L$.

$$\begin{aligned}
& \text{Diagram 1: A quadrilateral with vertices } c_1, c_2, c_3, r_1. \text{ Edges are } c_1c_2, c_2c_3, c_3r_1, r_1c_1. \text{ Diagonals are } c_1r_1 \text{ and } c_2r_1. \text{ Angles are } \beta \text{ at } c_1 \text{ and } \gamma \text{ at } r_1. \text{ A small red shaded region is at } c_3. \\
& = h_1 \text{ Diagram 2: Similar to Diagram 1, but the red shaded region is larger, covering the triangle } c_2c_3r_1. \text{ where } h_1 = Tri \begin{bmatrix} c_2 & c_3 & b \\ 1 & r_1 & 1 \end{bmatrix} \\
& = h_2 \text{ Diagram 3: Similar to Diagram 2, but the red shaded region is even larger, covering the triangle } c_1c_2r_1. \text{ where } h_2 = \lambda_{r_1}^{11} \\
& = h_3 \text{ Diagram 4: Similar to Diagram 3, but the red shaded region is the largest, covering the triangle } c_1c_2c_3. \text{ where } h_3 = h_2 Tri \begin{bmatrix} c_2 & b & c \\ c & \gamma & r_1 \end{bmatrix} \\
& = h_4 \text{ Diagram 5: Similar to Diagram 4, but the red shaded region is the largest, covering the triangle } c_1c_2c_3. \text{ Diagram 6: Similar to Diagram 5, but the red shaded region is the largest, covering the triangle } c_1c_2c_3. \text{ where } h_4 = h_3 Tri \begin{bmatrix} r_1 & \gamma & \beta \\ \gamma & c_2 & c \end{bmatrix} \\
& = h_5 \text{ Diagram 7: Similar to Diagram 6, but the red shaded region is the largest, covering the triangle } c_1c_2c_3. \text{ where } h_5 = \frac{\lambda_{r_1}^{11}}{\lambda_{c_2}^{c_1 1}} \\
& = h_5 Tet \begin{bmatrix} 1 & c_1 & 1 \\ c_2 & r_1 & \beta \end{bmatrix}
\end{aligned}$$

FIGURE 3.21. Detail of $RHSA_2R$.

(* Relation A₂ *)

```

lhsA2f[x_, y_, z_, a_, b_, c_, c1_] :=
  If[adm[a, x, 1] && adm[1, y, c1] && adm[c1, 1, b] && adm[c, z, 1],
    delta[c1] / theta[a, x, 1] / theta[1, y, c1] / theta[c1, 1, b] / theta[c, z, 1], 0];
lhsA2l[x_, y_, a_, b_, c1_] := trired[a, x, y, c1, x, 1] tet[a, b, 1, x, c1, a];
lhsA2r[y_, z_, b_, c_, c1_] := trired[z, z, 1, c1, c, y] tet[c1, b, c, z, c, 1];
lhsA2term[x_, y_, z_, a_, b_, c_, c1_] :=
  lhsA2l[x, y, a, b, c1] lhsA2r[y, z, b, c, c1] lhsA2f[x, y, z, a, b, c, c1];
lhsA2[x_, y_, z_, a_, b_, c_] :=
  Sum[lhsA2term[x, y, z, a, b, c, c1], {c1, y - 1, y + 1, 2}];

rhsA2f[x_, y_, z_, a_, b_, c_, l1_, c1_, c2_, c3_, r1_] :=
  If[adm[a, x, l1] && adm[l1, 1, 1] && adm[1, y, c1] && adm[c1, c2, 1] &&
    adm[c2, c3, 1] && adm[c3, 1, b] && adm[1, 1, r1] && adm[r1, z, c],
    delta[l1] delta[c1] delta[c2] delta[c3]
    delta[r1] / theta[a, x, l1] / theta[l1, 1, 1] / theta[1, y, c1] / theta[c1, c2, 1] /
    theta[c2, c3, 1] / theta[c3, 1, b] / theta[1, 1, r1] / theta[r1, z, c], 0];
rhsA2l[x_, y_, a_, b_, l1_, c1_, c2_, c3_] := trired[l1, 1, c2, c1, c3, 1]
  lambda[b, 1, c3] Sum[SixJ[a, x, 1, 1, i, l1] trired[x, y, 1, i, c1, x]
    trired[a, l1, c1, i, c3, x] tet[1, a, a, c3, b, i], {i, x - 1, x + 1, 2}];
rhsA2r[y_, z_, b_, c_, r1_, c1_, c2_, c3_] := lambda[c1, 1, c2] ^ -1
  lambda[1, 1, r1] ^ 2 trired[c2, c3, 1, r1, b, 1] trired[c2, b, c, z, c, r1]
  trired[r1, z, z, c2, y, c] tet[1, c1, c2, r1, 1, y];
rhsA2term[x_, y_, z_, a_, b_, c_, l1_, c1_, c2_, c3_, r1_] := rhsA2l[x, y, a, b, l1, c1,
  c2, c3] rhsA2r[y, z, b, c, r1, c1, c2, c3] rhsA2f[x, y, z, a, b, c, l1, c1, c2, c3, r1];
rhsA2[x_, y_, z_, a_, b_, c_] := Sum[rhsA2term[x, y, z, a, b, c, l1, c1, c2, c3, r1], {l1,
  0, 2, 2}, {r1, 0, 2, 2}, {c1, y - 1, y + 1, 2}, {c2, y - 2, y + 2, 2}, {c3, y - 3, y + 3, 2}];
A2[x_, y_, z_, a_, b_, c_] := If[adm[x, x, y] && adm[z, z, y] && adm[a, a, b] &&
  adm[c, c, b], (lhsA2[x, y, z, a, b, c] + A ^ (-3) rhsA2[x, y, z, a, b, c]) /
  norm[a, b, c], 0] // Simplify;

```

FIGURE 3.22. A₂ code in Mathematica.

3.8 Relation A_3

This is relation $A_3(\alpha, \beta, \gamma)$ and is not used in the text. We included the relation since $L(A_3(\alpha, \beta, \gamma)) = L(B_4(\alpha, \beta, \gamma))$ and this is reflected in the Mathematica code.

FIGURE 3.23. The left hand side of A_3 ($LHSA_3$).

FIGURE 3.24. The right hand side of A_3 ($RHSA_3$).

$$\frac{\delta_\alpha^a}{\Delta_\alpha \theta(1, \beta, c_1) \theta(c_1, 1, b) \theta(c, \gamma, 1)} = \frac{\delta_\alpha^a \Delta_{c_1}}{\Delta_\alpha \theta(1, \beta, c_1) \theta(c_1, 1, b) \theta(c, \gamma, 1)}$$

FIGURE 3.25. Detail of $LHSA_3F$.

$$\begin{aligned}
& \begin{array}{c} \text{Diagram 1} \end{array} \sim \begin{array}{c} \text{Diagram 2} \\ \\ = g_1 \begin{array}{c} \text{Diagram 3} \end{array} \text{ where } g_1 = \lambda_{c_1}^{b1} \\ \\ = g_{2,i} \begin{array}{c} \text{Diagram 4} \end{array} \text{ where } g_{2,i} = g_1 \sum_i \frac{\Delta_i}{(\lambda_i^{1\alpha})^2 \theta(1, \alpha, i)} \\ \\ = g_{3,i} \begin{array}{c} \text{Diagram 5} \end{array} \text{ where } g_{3,i} = g_{2,i} \text{Tri} \begin{bmatrix} i & 1 & c_1 \\ b & \alpha & \alpha \end{bmatrix} \\ \\ = g_{3,i} \text{Tet} \begin{bmatrix} 1 & i & c_1 \\ \alpha & \beta & \alpha \end{bmatrix}
\end{aligned}$$

FIGURE 3.26. Detail of $LHSA_3L$.

$$\begin{aligned}
& \begin{array}{c} \text{Diagram 1} \end{array} \\ \\ = h_1 \begin{array}{c} \text{Diagram 2} \end{array} \text{ where } h_1 = \text{Tri} \begin{bmatrix} c_1 & b & c \\ c & \gamma & 1 \end{bmatrix} \\ \\ = h_1 \text{Tet} \begin{bmatrix} 1 & c_1 & c \\ \gamma & \gamma & \beta \end{bmatrix}
\end{aligned}$$

FIGURE 3.27. Detail of $LHSA_3R$.

$$\begin{aligned}
& \frac{1}{\overline{a \begin{array}{|c|} \hline \alpha \\ \hline l_1 \quad l_2 \end{array} \beta \begin{array}{|c|} \hline c_1 \quad c_2 \quad c_3 \\ \hline \end{array} b \begin{array}{|c|} \hline \gamma \\ \hline r_1 \end{array} c}} \\
&= f_1 \frac{1}{\overline{\beta \begin{array}{|c|} \hline c_1 \quad c_2 \quad c_3 \\ \hline \end{array} b \begin{array}{|c|} \hline \gamma \\ \hline r_1 \end{array} c}} \quad \text{where } f_1 = \frac{\Delta_{l_1} \Delta_{l_2}}{\theta(a, 1, l_1) \theta(l_1, l_2, \alpha) \theta(l_2, 1, 1)} \\
&= f_2 \frac{1}{\overline{\begin{array}{|c|} \hline \gamma \\ \hline r_1 \end{array} c}} \quad \text{where } f_2 = f_1 \frac{\Delta_{c_1} \Delta_{c_2} \Delta_{c_3}}{\theta(1, \beta, c_1) \theta(c_1, c_2, 1) \theta(c_2, c_3, 1) \theta(c_3, 1, b)} \\
&= f_2 \frac{\Delta_{r_1}}{\theta(1, 1, r_1) \theta(r_1, \gamma, c)}
\end{aligned}$$

FIGURE 3.28. Detail of $RHSA_3F$.

$$\begin{aligned}
& \begin{array}{c} a \begin{array}{|c|} \hline l_1 \quad l_2 \\ \hline \alpha \quad \beta \\ \hline l_1 \quad l_2 \end{array} b \begin{array}{|c|} \hline c_3 \\ \hline c_2 \\ \hline c_1 \end{array} \end{array} = h_1 \quad \begin{array}{c} a \begin{array}{|c|} \hline l_1 \quad l_2 \\ \hline \alpha \quad \beta \\ \hline l_1 \quad l_2 \end{array} b \begin{array}{|c|} \hline c_3 \\ \hline c_2 \\ \hline c_1 \end{array} \end{array} \quad \text{where } h_1 = \text{Tri} \begin{bmatrix} l_2 & 1 & c_3 \\ c_2 & c_1 & 1 \end{bmatrix} \\
& \sim \begin{array}{c} a \begin{array}{|c|} \hline l_1 \quad l_2 \\ \hline \alpha \quad \beta \\ \hline l_1 \quad l_2 \end{array} b \begin{array}{|c|} \hline c_3 \\ \hline c_2 \\ \hline c_1 \end{array} \end{array} = h_2 \quad \begin{array}{c} a \begin{array}{|c|} \hline l_1 \quad l_2 \\ \hline \alpha \quad \beta \\ \hline l_1 \quad l_2 \end{array} b \begin{array}{|c|} \hline c_3 \\ \hline c_2 \\ \hline c_1 \end{array} \end{array} \quad \text{where } h_2 = h_1 \frac{\lambda_{c_3}^{b1}}{\lambda_{l_1}^{1a}} \\
& = h_3 \quad \begin{array}{c} a \begin{array}{|c|} \hline l_1 \quad l_2 \\ \hline \alpha \quad \beta \\ \hline l_1 \quad l_2 \end{array} b \begin{array}{|c|} \hline c_3 \\ \hline c_2 \\ \hline c_1 \end{array} \end{array} \quad \text{where } h_3 = h_2 \text{Tri} \begin{bmatrix} a & a & l_1 \\ 1 & c_3 & b \end{bmatrix} \\
& = h_{4,i} \quad \begin{array}{c} a \begin{array}{|c|} \hline l_1 \quad l_2 \\ \hline \alpha \quad \beta \\ \hline l_1 \quad l_2 \end{array} b \begin{array}{|c|} \hline c_3 \\ \hline c_2 \\ \hline c_1 \end{array} \end{array} \quad \text{where } h_{4,i} = h_3 \sum_i \frac{\Delta_i \lambda_i^{1\alpha}}{\theta(1, \alpha, i)} \\
& = h_{5,i} \quad \begin{array}{c} a \begin{array}{|c|} \hline l_1 \quad l_2 \\ \hline \alpha \quad \beta \\ \hline l_1 \quad l_2 \end{array} b \begin{array}{|c|} \hline c_3 \\ \hline c_2 \\ \hline c_1 \end{array} \end{array} \quad \text{where } h_{5,i} = h_{4,i} \text{Tri} \begin{bmatrix} l_1 & \alpha & i \\ 1 & 1 & l_2 \end{bmatrix}
\end{aligned}$$

FIGURE 3.29. Detail of $RHSA_3L$.

$$\begin{aligned}
&= h_{6,i,j} \quad j \quad \begin{array}{c} \text{Diagram 1: A graph with vertices } a, l_1, l_2, c_1, \alpha, \beta, i. \text{ Edges connect } a \text{ to } l_1, l_2, c_1; l_1 \text{ to } l_2, \alpha, \beta; l_2 \text{ to } \alpha, \beta; \alpha \text{ to } \beta, i; \beta \text{ to } i. \end{array} \sim j \quad \begin{array}{c} \text{Diagram 2: A graph with vertices } a, l_1, l_2, c_1, \alpha, \beta, i. \text{ Edges connect } a \text{ to } l_1, l_2, c_1; l_1 \text{ to } l_2, \alpha, \beta; l_2 \text{ to } \alpha, \beta; \alpha \text{ to } \beta, i; \beta \text{ to } i. \end{array} \quad \text{where } h_{6,i,j} = h_{5,i} \sum_j \frac{\Delta_j}{\theta(l_1, 1, j)} \\
&= h_{7,i,j} \quad j \quad \begin{array}{c} \text{Diagram 3: A graph with vertices } a, l_1, l_2, c_1, \alpha, \beta, i. \text{ Edges connect } a \text{ to } l_1, l_2, c_1; l_1 \text{ to } l_2, \alpha, \beta; l_2 \text{ to } \alpha, \beta; \alpha \text{ to } \beta, i; \beta \text{ to } i. \end{array} \quad \text{where } h_{7,i,j} = h_{6,i,j} \text{Tri} \begin{bmatrix} l_1 & 1 & j \\ l_1 & c_3 & a \end{bmatrix} \\
&= h_{8,i,j} \quad j \quad \begin{array}{c} \text{Diagram 4: A graph with vertices } a, l_1, l_2, c_1, \alpha, \beta, i. \text{ Edges connect } a \text{ to } l_1, l_2, c_1; l_1 \text{ to } l_2, \alpha, \beta; l_2 \text{ to } \alpha, \beta; \alpha \text{ to } \beta, i; \beta \text{ to } i. \end{array} \quad \text{where } h_{8,i,j} = h_{7,i,j} \text{Tri} \begin{bmatrix} \alpha & l_1 & j \\ c_3 & c_1 & l_2 \end{bmatrix} \\
&= h_{9,i,j} \quad j \quad \begin{array}{c} \text{Diagram 5: A graph with vertices } a, l_1, l_2, c_1, \alpha, \beta, i. \text{ Edges connect } a \text{ to } l_1, l_2, c_1; l_1 \text{ to } l_2, \alpha, \beta; l_2 \text{ to } \alpha, \beta; \alpha \text{ to } \beta, i; \beta \text{ to } i. \end{array} \quad \text{where } h_{9,i,j} = h_{8,i,j} \text{Tri} \begin{bmatrix} \alpha & \alpha & j \\ c_1 & 1 & \beta \end{bmatrix} \\
&= h_{9,i,j} \text{Tet} \begin{bmatrix} 1 & \alpha & j \\ 1 & l_1 & i \end{bmatrix}
\end{aligned}$$

FIGURE 3.30. Detail of $RHSA_3L$ continued.

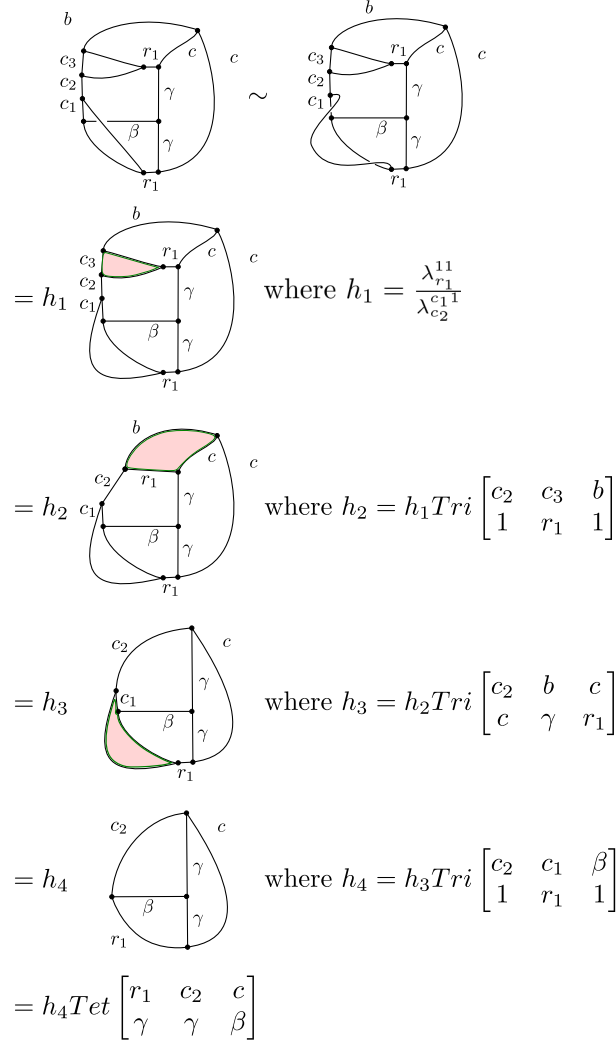


FIGURE 3.31. Detail of RHA_3R .

(* Left Hand Side of Relation A_3 *)

```

lhsA3f[x_, y_, z_, b_, c_, c1_] := If[ adm[1, y, c1] && adm[c1, 1, b] && adm[c, z, 1],
  delta[c1]/delta[x]/theta[1, y, c1]/theta[c1, 1, b]/theta[c, z, 1], 0];
lhsA3l[x_, y_, b_, c1_] := lambda[b, 1, c1]
Sum[If[adm[1, x, i], lambda[1, x, i]^2 (delta[i]/theta[1, x, i])
  tired[i, 1, b, x, c1, x] tet[1, i, x, y, c1, x], 0], {i, x-1, x+1, 2}];
lhsA3r[y_, z_, b_, c_, c1_] := tired[c1, b, c, z, c, 1] tet[1, c1, z, z, c, y];
lhsA3term[x_, y_, z_, b_, c_, c1_] :=
  lhsA3l[x, y, b, c1] lhsA3r[y, z, b, c, c1] lhsA3f[x, y, z, b, c, c1];
lhsA3[x_, y_, z_, a_, b_, c_] :=
  If[x == a, Sum[lhsA3term[x, y, z, b, c, c1], {c1, y-1, y+1, 2}], 0];

```

FIGURE 3.32. A_3 code in Mathematica.

3.9 Relation B_4

In this relation, the left hand side is the left hand side of relation A_3 rotated by π . In the evaluation of the left hand side of B_4 , we have $\text{lhs}B_4[x, y, z, a, b, c] = \text{lhs}A_3[z, y, x, c, b, a]$ in our code.

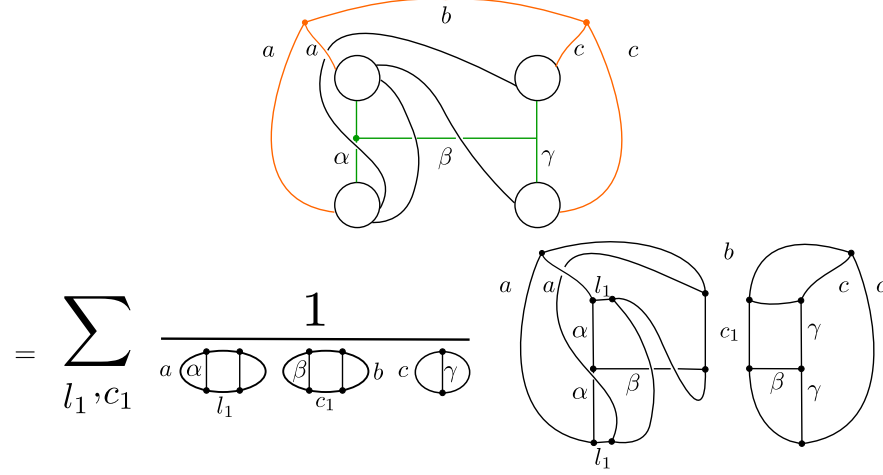


FIGURE 3.33. The right hand side of B_4 ($RHSB_4$).

$$\frac{1}{\text{graph with nodes } a, b, c \text{ and internal nodes } \alpha, \beta, \gamma, l_1, c_1}} = \frac{\Delta_{l_1} \Delta_{c_1}}{\theta(a, \alpha, l_1) \theta(l_1, 1, 1) \theta(1, \beta, c_1) \theta(c_1, 1, b) \theta(c, \gamma, 1)}$$

FIGURE 3.34. Detail of $RHSB_4F$.

$$\begin{aligned}
& \text{Diagram 1} = h_1 \text{ Diagram 2} \quad \text{where } h_1 = \frac{1}{\lambda_{c_1}^{\beta 1}} \\
& = h_{2,i} \text{ Diagram 3} \quad \text{where } h_{2,i} = h_1 \sum_i \frac{\Delta_i \lambda_i^{1\beta}}{\theta(1,\beta,i)} \\
& = h_{3,i} \text{ Diagram 4} \quad \text{where } h_{3,i} = h_{2,i} \text{Tri} \begin{bmatrix} i & 1 & l_1 \\ 1 & c_1 & \beta \end{bmatrix} \\
& = h_{4,i,j} \text{ Diagram 5} \quad \text{where } h_{4,i,j} = h_{3,i} \sum_j \frac{\Delta_j \lambda_j^{1\alpha}}{\theta(1,\alpha,j)} \\
& = h_{5,i,j} \text{ Diagram 6} \quad \text{where } h_{5,i,j} = h_{4,i,j} \text{Tri} \begin{bmatrix} a & \alpha & j \\ 1 & 1 & l_1 \end{bmatrix} \\
& = h_{6,i,j,k} \text{ Diagram 7} \quad \text{where } h_{6,i,j,k} = h_{5,i,j} \sum_k \frac{\Delta_k \lambda_k^{a1}}{\theta(a,1,k)} \\
& = h_{7,i,j,k} \text{ Diagram 8} \quad \text{where } h_{7,i,j,k} = h_{6,i,j,k} \text{Tri} \begin{bmatrix} a & b & c_1 \\ 1 & k & a \end{bmatrix}
\end{aligned}$$

FIGURE 3.35. Detail of $RHSB_4L$.

$$\begin{aligned}
&= h_{8,i,j,k,m} \quad \text{[Diagram: A complex polygon with vertices labeled } a, k, c_1, l_1, i, j, a. \text{ Internal lines connect } a \text{ to } k, k \text{ to } c_1, c_1 \text{ to } l_1, l_1 \text{ to } i, i \text{ to } j, j \text{ to } a. \text{ A shaded region is formed by } a, k, c_1. \text{]} \sim \text{[Diagram: A complex polygon with vertices labeled } a, k, c_1, l_1, i, j, a. \text{ Internal lines connect } a \text{ to } k, k \text{ to } c_1, c_1 \text{ to } l_1, l_1 \text{ to } i, i \text{ to } j, j \text{ to } a. \text{ A shaded region is formed by } a, k, c_1. \text{]} \quad \text{where } h_{8,i,j,k,m} = h_{7,i,j,k} \sum_m \frac{\Delta_m}{\theta(a,1,m)} \\
&= h_{9,i,j,k,m} \quad \text{[Diagram: A complex polygon with vertices labeled } a, l_1, c_1, i, j, a. \text{ Internal lines connect } a \text{ to } l_1, l_1 \text{ to } c_1, c_1 \text{ to } i, i \text{ to } j, j \text{ to } a. \text{ A shaded region is formed by } a, l_1, c_1. \text{]} \quad \text{where } h_{9,i,j,k,m} = h_{8,i,j,k,m} \text{Tri} \begin{bmatrix} a & 1 & m \\ a & c_1 & k \end{bmatrix} \\
&= h_{10,i,j,k,m} \quad \text{[Diagram: A complex polygon with vertices labeled } \alpha, l_1, i, j, a. \text{ Internal lines connect } \alpha \text{ to } l_1, l_1 \text{ to } i, i \text{ to } j, j \text{ to } a. \text{ A shaded region is formed by } \alpha, l_1, i. \text{]} \quad \text{where } h_{10,i,j,k,m} = h_{9,i,j,k,m} \text{Tri} \begin{bmatrix} \alpha & a & m \\ c_1 & i & l_1 \end{bmatrix} \\
&= h_{11,i,j,k,m} \quad \text{[Diagram: A complex polygon with vertices labeled } \alpha, j, a. \text{ Internal lines connect } \alpha \text{ to } j, j \text{ to } a. \text{ A shaded region is formed by } \alpha, j, a. \text{]} \quad \text{where } h_{11,i,j,k,m} = h_{10,i,j,k,m} \text{Tri} \begin{bmatrix} \alpha & \alpha & m \\ i & 1 & \beta \end{bmatrix} \\
&= h_{11,i,j,k,m} \text{Tet} \begin{bmatrix} 1 & \alpha & m \\ 1 & a & j \end{bmatrix}
\end{aligned}$$

FIGURE 3.36. Detail of $RHSB_4L$ continued.

$$\begin{aligned}
&\text{[Diagram: A complex polygon with vertices labeled } b, c, c_1, \gamma, \beta, \gamma. \text{ Internal lines connect } b \text{ to } c, c \text{ to } c_1, c_1 \text{ to } \gamma, \gamma \text{ to } \beta, \beta \text{ to } \gamma. \text{ A shaded region is formed by } b, c, c_1. \text{]} \\
&= h_1 \quad \text{[Diagram: A complex polygon with vertices labeled } c, c_1, \gamma, \beta, \gamma. \text{ Internal lines connect } c \text{ to } c_1, c_1 \text{ to } \gamma, \gamma \text{ to } \beta, \beta \text{ to } \gamma. \text{ A shaded region is formed by } c, c_1, \gamma. \text{]} \quad \text{where } h_1 = \frac{\text{Tet} \begin{bmatrix} c_1 & b & c \\ c & \gamma & 1 \end{bmatrix}}{\theta(c, c_1, \gamma)} \\
&= h_1 \text{Tet} \begin{bmatrix} 1 & c_1 & c \\ \gamma & \gamma & \beta \end{bmatrix}
\end{aligned}$$

FIGURE 3.37. Detail of $RHSB_4R$.


```

(* Relation B4 *)

lhsB4[x_, y_, z_, a_, b_, c_] := lhsA3[z, y, x, c, b, a];
rhsB4f[x_, y_, z_, a_, b_, c_, l1_, c1_] :=
  If[adm[a, x, l1] && adm[l1, 1, 1] && adm[1, y, c1] && adm[c1, 1, b] && adm[c, z, 1],
    delta[l1] delta[c1] / theta[a, x, l1] / theta[l1, 1, 1] / theta[1, y, c1] /
    theta[c1, 1, b] / theta[c, z, 1], 0];
rhsB4l[x_, y_, a_, b_, l1_, c1_] := lambda[y, 1, c1] ^ -1
  Sum[If[adm[1, y, i] && adm[1, x, j] && adm[a, 1, k] && adm[a, 1, m],
    (delta[i] / theta[1, y, i]) lambda[1, y, i] trired[i, 1, 1, c1, l1, y]
    (delta[j] / theta[1, x, j]) lambda[1, x, j] trired[a, x, 1, 1, j, l1]
    (delta[k] / theta[a, 1, k]) lambda[a, 1, k] trired[a, b, 1, k, c1, a]
    (delta[m] / theta[a, 1, m]) trired[a, 1, a, c1, m, k]
    trired[x, a, c1, i, m, l1] trired[x, x, i, 1, m, y] tet[1, x, 1, a, m, j], 0],
    {i, y - 1, y + 1, 2}, {j, x - 1, x + 1, 2}, {k, a - 1, a + 1, 2}, {m, a - 1, a + 1, 2}];
rhsB4r[y_, z_, b_, c_, c1_] := trired[c1, b, c, z, c, 1] tet[1, c1, z, z, c, y];
rhsB4term[x_, y_, z_, a_, b_, c_, l1_, c1_] :=
  rhsB4f[x, y, z, a, b, c, l1, c1] rhsB4l[x, y, a, b, l1, c1] rhsB4r[y, z, b, c, c1];
rhsB4[x_, y_, z_, a_, b_, c_] := Sum[rhsB4term[x, y, z, a, b, c, l1, c1],
  {l1, 0, 2, 2}, {c1, y - 1, y + 1, 2}];
B4[x_, y_, z_, a_, b_, c_] :=
  If[adm[x, x, y] && adm[z, z, y] && adm[a, a, b] && adm[c, c, b],
    (lhsB4[x, y, z, a, b, c] - rhsB4[x, y, z, a, b, c]) / norm[a, b, c], 0] // Simplify;

```

FIGURE 3.38. B_4 code in Mathematica.

Appendix B: Mathematica Code

John Harris' Code with modification to tet evaluation by simplifying min and max using given conditions.
 oddq[] and evenq[] extend Oddq[] and Evenq[] to variables.

```
In[1]:= oddq[a_ b_ /; oddq[a] && oddq[b]] := True;
oddq[a_ + b_ /; (oddq[a] && evenq[b]) || (evenq[a] && oddq[b])] := True;
oddq[a_] := OddQ[a];

evenq[a_ b_ /; (evenq[a] && IntegerQ[b]) || (evenq[b] && IntegerQ[a])] := True;
evenq[a_ + b_ /; (evenq[a] && evenq[b]) || (oddq[a] && oddq[b])] := True;
evenq[a_] := EvenQ[a];
```

quantum integers and their factorials are left unevaluated. lambdas, thetas, and tets are evaluated as in Kauffman-Lins.

```
In[7]:= qi[0] = 0; qi[1] = 1;

qif[0] = 1; qif[n_ /; n ≥ 1] := qif[n - 1] qi[n];
qif[n_ + x_ /; n ≥ 1] := qif[n + x - 1] qi[n + x];

delta[n_] := (-1) ^ n qi[n + 1];

adm[a1_, b1_, c1_] := Module[{a = Simplify[a1], b = Simplify[b1], c = Simplify[c1]},
  Simplify[a ≥ 0 && b ≥ 0 && c ≥ 0 && Abs[a - b] ≤ c && c ≤ a + b, given] && evenq[a + b + c]];

lambda[a_, b_, c_] := (-1) ^ ((a + b - c) / 2) A ^ ((a (a + 2) + b (b + 2) - c (c + 2)) / 2);

theta[a_, b_, c_] := Module[{
  m = (a + b - c) / 2 // Simplify,
  n = (b + c - a) / 2 // Simplify,
  p = (a + c - b) / 2 // Simplify
},
  If[adm[a, b, c],
    (-1) ^ (m + n + p) qif[m + n + p + 1] qif[m]
    qif[n] qif[p] / qif[m + n] / qif[n + p] / qif[m + p] // Simplify,
    0]
];
```

```

In[14]:= admtet[a_, b_, c_, d_, e_, f_] :=
  adm[a, d, e] && adm[b, c, e] && adm[a, b, f] && adm[c, d, f];

tet[a_, b_, c_, d_, e_, f_] := Module[{
  a1 = (a + d + e) / 2 // Simplify,
  a2 = (b + c + e) / 2 // Simplify,
  a3 = (a + b + f) / 2 // Simplify,
  a4 = (c + d + f) / 2 // Simplify,
  av,

  b1 = (b + d + e + f) / 2 // Simplify,
  b2 = (a + c + e + f) / 2 // Simplify,
  b3 = (a + b + c + d) / 2 // Simplify,
  bv,

  m, M, cv, s
},

  av = {a1, a2, a3, a4}; bv = {b1, b2, b3};

  m = Simplify[Max[a1, a2, a3, a4], given];
  M = Simplify[Min[b1, b2, b3], given];

  If[admtet[a, b, c, d, e, f],
    intfac = Product[qif[bv[[j]] - av[[i]]], {i, 1, 4}, {j, 1, 3}];
    extfac = qif[a] qif[b] qif[c] qif[d] qif[e] qif[f];
    cv = Intersection[av, bv];
    (intfac / extfac) If[Length[cv] > 0, s = cv[[1]];
      (-1)^s qif[s + 1] / Product[qif[s - av[[i]]], {i, 1, 4}] / Product[qif[bv[[j]] -
        s], {j, 1, 3}], Sum[(-1)^s qif[s + 1] / Product[qif[s - av[[i]]], {i, 1,
        4}] / Product[qif[bv[[j]] - s], {j, 1, 3}], {s, m, M}]] // Simplify,
    0]
  ];

norm[a_, b_, c_] := theta[a, a, b] theta[b, c, c] / delta[a] / delta[b] / delta[c];

```

This marks the end of John Harris' code. The following functions *trired* and *SixJ* are defined in the same way as Susan Abernathy's code.

```

In[17]:= trired[a_, b_, c_, d_, e_, f_] :=
  If[admtet[a, b, c, d, e, f], tet[a, b, c, d, e, f] / theta[a, d, e], 0];

SixJ[a_, b_, c_, d_, e_, f_] := If[admtet[a, b, c, d, e, f],
  tet[a, b, c, d, e, f] delta[e] / theta[a, d, e] / theta[b, c, e], 0];

```

Let REL denote relation, then we have:

lhsREL = Left hand side of the relation.

rhsREL = Right hand side of the relation.

with

rhsRELf := The factors coming from three 2-sphere reductions in the connected sum

$S^1 \times S^2 \# S^1 \times S^2$.

rhsRELI := The left side of the diagram after 2-sphere reductions.

rhsRELr := The right side of the diagram after 2-sphere reductions.

rhsRELterm := (rhsRELf)(rhsRELI)(rhsRELr).

rhsREL := the sum of rhsRELterm over admissible colorings after the three 2-sphere reductions for a given $g_{(x,y,z)}$ and $g_{(a,b,c)}$.

norm[a,b,c] := $\langle g_{(a,b,c)}, g_{(a,b,c)} \rangle$.

RELN := (lhsREL - rhsREL) / $\langle g_{(a,b,c)}, g_{(a,b,c)} \rangle$.

```
In[19]:= (* Relation A1*)
```

```
lhsA1[x_, y_, z_, a_, b_, c_] := If[adm[a, x, 1] && c == z && b == y,
  theta[z, z, y] tet[x, a, a, x, y, 1] / delta[y] / delta[z] / theta[a, x, 1],
  0];
rhsA1[x_, y_, z_, a_, b_, c_] := lhsA1[z, y, x, c, b, a];
A1[x_, y_, z_, a_, b_, c_] :=
  If[adm[x, x, y] && adm[z, z, y] && adm[a, a, b] && adm[c, c, b],
    (lhsA1[x, y, z, a, b, c] - rhsA1[x, y, z, a, b, c]) / norm[a, b, c], 0] // Simplify;
```

```
(* Relation A2 *)
```

```
lhsA2f[x_, y_, z_, a_, b_, c_, c1_] :=
  If[adm[a, x, 1] && adm[1, y, c1] && adm[c1, 1, b] && adm[c, z, 1],
    delta[c1] / theta[a, x, 1] / theta[1, y, c1] / theta[c1, 1, b] / theta[c, z, 1], 0];
lhsA2l[x_, y_, a_, b_, c1_] := trired[a, x, y, c1, x, 1] tet[a, b, 1, x, c1, a];
lhsA2r[y_, z_, b_, c_, c1_] := trired[z, z, 1, c1, c, y] tet[c1, b, c, z, c, 1];
lhsA2term[x_, y_, z_, a_, b_, c_, c1_] :=
  lhsA2l[x, y, a, b, c1] lhsA2r[y, z, b, c, c1] lhsA2f[x, y, z, a, b, c, c1];
lhsA2[x_, y_, z_, a_, b_, c_] :=
  Sum[lhsA2term[x, y, z, a, b, c, c1], {c1, y - 1, y + 1, 2}];

rhsA2f[x_, y_, z_, a_, b_, c_, l1_, c1_, c2_, c3_, r1_] :=
  If[adm[a, x, l1] && adm[l1, 1, 1] && adm[1, y, c1] && adm[c1, c2, 1] &&
    adm[c2, c3, 1] && adm[c3, 1, b] && adm[1, 1, r1] && adm[r1, z, c],
    delta[l1] delta[c1] delta[c2] delta[c3]
    delta[r1] / theta[a, x, l1] / theta[l1, 1, 1] / theta[1, y, c1] / theta[c1, c2, 1] /
    theta[c2, c3, 1] / theta[c3, 1, b] / theta[1, 1, r1] / theta[r1, z, c], 0];
rhsA2l[x_, y_, a_, b_, l1_, c1_, c2_, c3_] := trired[l1, 1, c2, c1, c3, 1]
  lambda[b, 1, c3] Sum[SixJ[a, x, 1, 1, i, l1] trired[x, y, 1, i, c1, x]
    trired[a, l1, c1, i, c3, x] tet[1, a, a, c3, b, i], {i, x - 1, x + 1, 2}];
rhsA2r[y_, z_, b_, c_, r1_, c1_, c2_, c3_] := lambda[c1, 1, c2] ^ -1
  lambda[1, 1, r1] ^ 2 trired[c2, c3, 1, r1, b, 1] trired[c2, b, c, z, c, r1]
  trired[r1, z, z, c2, y, c] tet[1, c1, c2, r1, 1, y];
rhsA2term[x_, y_, z_, a_, b_, c_, l1_, c1_, c2_, c3_, r1_] := rhsA2l[x, y, a, b, l1, c1,
  c2, c3] rhsA2r[y, z, b, c, r1, c1, c2, c3] rhsA2f[x, y, z, a, b, c, l1, c1, c2, c3, r1];
rhsA2[x_, y_, z_, a_, b_, c_] := Sum[rhsA2term[x, y, z, a, b, c, l1, c1, c2, c3, r1], {l1,
  0, 2, 2}, {r1, 0, 2, 2}, {c1, y - 1, y + 1, 2}, {c2, y - 2, y + 2, 2}, {c3, y - 3, y + 3, 2}];
A2[x_, y_, z_, a_, b_, c_] := If[adm[x, x, y] && adm[z, z, y] && adm[a, a, b] &&
  adm[c, c, b], (lhsA2[x, y, z, a, b, c] + A ^ (-3) rhsA2[x, y, z, a, b, c]) /
  norm[a, b, c], 0] // Simplify;
```

In[28]:= (* Left Hand Side of Relation A₃ *)

```
lhsA3f[x_, y_, z_, b_, c_, c1_] := If[ adm[1, y, c1] && adm[c1, 1, b] && adm[c, z, 1],
  delta[c1] / delta[x] / theta[1, y, c1] / theta[c1, 1, b] / theta[c, z, 1], 0];
lhsA3l[x_, y_, b_, c1_] := lambda[b, 1, c1]
  Sum[If[adm[1, x, i], lambda[1, x, i]^-2 (delta[i] / theta[1, x, i])
    trired[i, 1, b, x, c1, x] tet[1, i, x, y, c1, x], 0], {i, x-1, x+1, 2}];
lhsA3r[y_, z_, b_, c_, c1_] := trired[c1, b, c, z, c, 1] tet[1, c1, z, z, c, y];
lhsA3term[x_, y_, z_, b_, c_, c1_] :=
  lhsA3l[x, y, b, c1] lhsA3r[y, z, b, c, c1] lhsA3f[x, y, z, b, c, c1];
lhsA3[x_, y_, z_, a_, b_, c_] :=
  If[x == a, Sum[lhsA3term[x, y, z, b, c, c1], {c1, y-1, y+1, 2}], 0];
```

(* Relation B₄ *)

```
lhsB4[x_, y_, z_, a_, b_, c_] := lhsA3[z, y, x, c, b, a];
rhsB4f[x_, y_, z_, a_, b_, c_, l1_, c1_] :=
  If[adm[a, x, l1] && adm[l1, 1, 1] && adm[1, y, c1] && adm[c1, 1, b] && adm[c, z, 1],
    delta[l1] delta[c1] / theta[a, x, l1] / theta[l1, 1, 1] / theta[1, y, c1] /
      theta[c1, 1, b] / theta[c, z, 1], 0];
rhsB4l[x_, y_, a_, b_, l1_, c1_] := lambda[y, 1, c1]^-1
  Sum[If[adm[1, y, i] && adm[1, x, j] && adm[a, 1, k] && adm[a, 1, m],
    (delta[i] / theta[1, y, i]) lambda[1, y, i] trired[i, 1, 1, c1, l1, y]
    (delta[j] / theta[1, x, j]) lambda[1, x, j] trired[a, x, 1, 1, j, l1]
    (delta[k] / theta[a, 1, k]) lambda[a, 1, k] trired[a, b, 1, k, c1, a]
    (delta[m] / theta[a, 1, m]) trired[a, 1, a, c1, m, k]
    trired[x, a, c1, i, m, l1] trired[x, x, i, 1, m, y] tet[1, x, 1, a, m, j], 0],
    {i, y-1, y+1, 2}, {j, x-1, x+1, 2}, {k, a-1, a+1, 2}, {m, a-1, a+1, 2}];
rhsB4r[y_, z_, b_, c_, c1_] := trired[c1, b, c, z, c, 1] tet[1, c1, z, z, c, y];
rhsB4term[x_, y_, z_, a_, b_, c_, l1_, c1_] :=
  rhsB4f[x, y, z, a, b, c, l1, c1] rhsB4l[x, y, a, b, l1, c1] rhsB4r[y, z, b, c, c1];
rhsB4[x_, y_, z_, a_, b_, c_] :=
  Sum[rhsB4term[x, y, z, a, b, c, l1, c1], {l1, 0, 2, 2}, {c1, y-1, y+1, 2}];
B4[x_, y_, z_, a_, b_, c_] :=
  If[adm[x, x, y] && adm[z, z, y] && adm[a, a, b] && adm[c, c, b],
    (lhsB4[x, y, z, a, b, c] - rhsB4[x, y, z, a, b, c]) / norm[a, b, c], 0] // Simplify;
```

We choose to leave theta, tet, lambda, delta unevaluated until the end of calculations and call the code below when we need to.

Evaluation of quantum integers and quantum integer factorials.

```
In[40]:= qfev[e_] :=
```

```
ReplaceRepeated[ e // Expand, {qi[n_] → (A^(2 n) - A^(-2 n)) / (A^2 - A^(-2)),
  qif[x] → qi[x] qif[x - 1], qif[x - 1] → qi[x - 1] qif[x - 2],
  qif[x - 2] → qi[x - 2] qif[x - 3], qif[x - 3] → qi[x - 3] qif[x - 4],
  qif[x - 4] → qi[x - 4] qif[x - 5], qif[x - 5] → qi[x - 5] qif[x - 6],
  qif[x - 6] → qi[x - 6] qif[x - 7], qif[x - 7] → qi[x - 7] qif[x - 8],
  qif[x - 8] → qi[x - 8] qif[x - 9], qif[y] → qi[y] qif[y - 1],
  qif[y - 1] → qi[y - 1] qif[y - 2], qif[y - 2] → qi[y - 2] qif[y - 3],
  qif[y - 3] → qi[y - 3] qif[y - 4], qif[y - 4] → qi[y - 4] qif[y - 5],
  qif[y - 5] → qi[y - 5] qif[y - 6], qif[y - 6] → qi[y - 6] qif[y - 7], qif[y - 7] →
  qi[y - 7] qif[y - 8], qif[z] → qi[z] qif[z - 1], qif[z - 1] → qi[z - 1] qif[z - 2],
  qif[z - 2] → qi[z - 2] qif[z - 3], qif[z - 3] → qi[z - 3] qif[z - 4],
  qif[ $\frac{y}{2}$ ] → qi[ $\frac{y}{2}$ ] qif[ $\frac{y}{2} - 1$ ], qif[ $\frac{y}{2} - 1$ ] → qi[ $\frac{y}{2} - 1$ ] qif[ $\frac{y}{2} - 2$ ],
  qif[ $\frac{y}{2} - 2$ ] → qi[ $\frac{y}{2} - 2$ ] qif[ $\frac{y}{2} - 3$ ], qif[ $\frac{y}{2} - 3$ ] → qi[ $\frac{y}{2} - 3$ ] qif[ $\frac{y}{2} - 4$ ],
  qif[ $\frac{y}{2} - 4$ ] → qi[ $\frac{y}{2} - 4$ ] qif[ $\frac{y}{2} - 5$ ], qif[ $x - \frac{y}{2}$ ] → qi[ $x - \frac{y}{2}$ ] qif[ $x - \frac{y}{2} - 1$ ],
  qif[ $z + \frac{y}{2}$ ] → qi[ $z + \frac{y}{2}$ ] qif[ $z + \frac{y}{2} - 1$ ], qif[ $z + \frac{y}{2} - 1$ ] → qi[ $z + \frac{y}{2} - 1$ ] qif[ $z + \frac{y}{2} - 2$ ],
  qif[ $z + \frac{y}{2} - 2$ ] → qi[ $z + \frac{y}{2} - 2$ ] qif[ $z + \frac{y}{2} - 3$ ],
  qif[ $z + \frac{y}{2} - 3$ ] → qi[ $z + \frac{y}{2} - 3$ ] qif[ $z + \frac{y}{2} - 4$ ], qif[ $z + \frac{y}{2} - 4$ ] →
  qi[ $z + \frac{y}{2} - 4$ ] qif[ $z + \frac{y}{2} - 5$ ], qif[ $z + \frac{y}{2} - 5$ ] → qi[ $z + \frac{y}{2} - 5$ ] qif[ $z + \frac{y}{2} - 6$ ],
  qif[ $x + \frac{y}{2}$ ] → qi[ $x + \frac{y}{2}$ ] qif[ $x + \frac{y}{2} - 1$ ], qif[ $x + \frac{y}{2} - 1$ ] → qi[ $x + \frac{y}{2} - 1$ ] qif[ $x + \frac{y}{2} - 2$ ],
  qif[ $x + \frac{y}{2} - 2$ ] → qi[ $x + \frac{y}{2} - 2$ ] qif[ $x + \frac{y}{2} - 3$ ], qif[ $x + \frac{y}{2} - 3$ ] →
  qi[ $x + \frac{y}{2} - 3$ ] qif[ $x + \frac{y}{2} - 4$ ], qif[ $x + \frac{y}{2} - 4$ ] → qi[ $x + \frac{y}{2} - 4$ ] qif[ $x + \frac{y}{2} - 5$ ],
  qif[ $x + \frac{y}{2} - 5$ ] → qi[ $x + \frac{y}{2} - 5$ ] qif[ $x + \frac{y}{2} - 6$ ], qif[ $x + \frac{y}{2} - 6$ ] →
  qi[ $x + \frac{y}{2} - 6$ ] qif[ $x + \frac{y}{2} - 7$ ], qif[ $x - \frac{y}{2} - 1$ ] → qi[ $x - \frac{y}{2} - 1$ ] qif[ $x - \frac{y}{2} - 2$ ],
  qif[ $\frac{1}{2}(-8 + y)$ ] → qif[ $-4 + \frac{y}{2}$ ], qif[ $\frac{1}{2}(-4 + y)$ ] → qif[ $-2 + \frac{y}{2}$ ],
  qif[ $\frac{1}{2}(-6 + y)$ ] → qif[ $-3 + \frac{y}{2}$ ]}] // Expand // Factor;
```

Coefficient for highest term in relation $B_4(x-2, y-2, z-1)$. The conditions are $x \geq 2$, $y \geq 4$, $z \geq 1$, $2x-2 \geq y$, and $2z \geq y$.


```
In[41]:= IntegerQ[x] ^= True; EvenQ[y] ^= True; IntegerQ[z] ^= True;
given = x ≥ 2 && y ≥ 4 && z ≥ 1 && 2 x - 2 ≥ y && 2 z ≥ y;
B4[x - 2, y - 2, z - 1, x, y, z] // qfev // FullSimplify
```

Out[43]= $-A^{-2x+2y}$

Coefficient for the highest term in relation $B_4(x-2, y-2, z-1)$ when $y=2$.

```
In[44]:= B4[x - 2, 0, z - 1, x, 2, z] // qfev // FullSimplify
```

Out[44]= $-A^{4-2x}$

Coefficient for highest term in relation $A_2(x-2, y-4, z-2)$ to rewrite (x, y, z) . The conditions are $x \geq 2$, $y \geq 4$, $z \geq 2$, $2x \geq y$, and $2z \geq y$.

```
In[45]:= IntegerQ[x] ^= True; EvenQ[y] ^= True; IntegerQ[z] ^= True;
given = x ≥ 2 && y ≥ 4 && z ≥ 2 && y ≤ 2 x && y ≤ 2 z;
A2[x - 2, y - 4, z - 2, x, y, z] // qfev
```

Out[47]= $-A^{-6+2y}$

When we replace the above with $y = 2x$ we have:

```
In[48]:= (A2[x - 2, y - 4, z - 2, x, y, z] // qfev) /. y → 2 x
```

Out[48]= $-A^{-6+4x}$

Rewriting $(1, 2, z)$ using relation $A_1(1, 2, z-1)$ for $z \geq 3$.

```
In[49]:= IntegerQ[z] ^= True; given = z ≥ 3;
```

```
In[50]:= A1[1, 2, z - 1, 1, 2, z] // qfev
```

Out[50]= -1

```
In[51]:= A1[1, 2, z - 1, 2, 2, z - 1] // qfev
```

Out[51]= 1

```
In[52]:= A1[1, 2, z - 1, 1, 2, z - 2] // qfev // Simplify
```

Out[52]=
$$\frac{-A^8 + A^{4z} + A^{4(3+z)} - A^{4+8z}}{A^4 (A^4 - A^{4z} + A^{8z} - A^{4+4z})}$$

Rewriting $(1, 2, 2)$ as $(2, 2, 1)$ using relation A_1 (the case when $z = 2$).

```
In[53]:= A1[1, 2, 1, 1, 2, 2] // qfev
```

Out[53]= -1

```
In[54]:= A1[1, 2, 1, 2, 2, 1] // qfev
```

Out[54]= 1

Vita

Andrew R. Holmes was born in Plattsburgh, New York and grew up in Edmond, Oklahoma. He finished his undergraduate studies at the University of Oklahoma in May 2012. In August 2012, he came to Louisiana State University to pursue graduate studies in mathematics as a graduate teaching assistant. He earned a master of science degree in mathematics from Louisiana State University in May 2014. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in May 2017.